Math 28: The Theory of Error-Correcting Codes

Positivity of \( N_{m+1} \) for extremal enumerators

Given power series \( F = F_0q^t + O(q^2) \) and \( \Delta = q + O(q^2) \) and an integer \( a \in [0, t) \), for every positive integer \( m \) there is a unique power series \( W_m = 1 + O(q^{m+1}) \) of the form \( W_m = F_0q^t P(F, \Delta) \) where \( P_m \) is a homogeneous polynomial of degree \( m \). We are interested in the coefficients \( N_{m+1}, N_{m+2}, \ldots \) of \( W_m \). We begin with the following more-or-less elementary formula for \( N_{m+1} \):

**Lemma 1.** \( N_{m+1} \) is \( (tm+a)/(tm+t) \) times the \( q^{-1} \) coefficient of \( F_0^{m-a} F^t/\Delta^{m+1} = aF_0q^{t-a}/\Delta^{m+1} \), where \( F^t = dF/dq \).

For example, for the extended binary Golay code \( \mathcal{G}_{24} \) we have \( t = 3, F_0 = 1 + 14q + q^2, a = 0, \) and \( m = 1 \), and calculate that

\[
\frac{F_0^{-a} F'(q)}{\Delta^{m+1}} = \frac{(42 + 6q)F_0^2}{\Delta^2} = 42q^2 + 1518q^{-1} + 19452 + 117828q + \cdots
\]

and indeed \( \frac{3}{6} \times 1518 = 750 \).

**Proof of Lemma 1:** Let \( z = \Delta/F = q + O(q^2) \). Then \( W_m/(F_0^m F^m) = P(1, z) \), so if we let

\[
\frac{1}{F_0^m F^m} = \sum_{k=0}^{\infty} b_k z^k
\]

be the formal expansion of \( 1/(F_0^m F^m) \) in powers of \( z \) then \( P(1, z) = \sum_{k=0}^{m} b_k z^k \). Therefore

\[
1 - W_m = \sum_{k=m+1}^{\infty} b_k z^k.
\]

and in particular \( N_{m+1} \) is just \(-b_{m+1}\). We isolate this coefficient using *invariance of the residue* under locally invertible change of coordinate; that’s a result that’s usually introduced in complex analysis but it can be proved also by formal power-series manipulation.\(^1\) We find that \(-b_{m+1}\) is

\[
- \text{Res} \left( \frac{1}{F_0^m F^m} \frac{dz}{z^{m+2}} \right) = - \text{Res} \left( \frac{(\Delta/F)_m d(\Delta/F)}{F_0^m F^m (\Delta/F)^2} \right) = + \text{Res} \left( \frac{\Delta dF - F d\Delta}{F_0^m \Delta^{m+2}} \right).
\]

But the \( F d\Delta \) and \( \Delta dF \) parts of this formula are related, because \( F_0^a = F^{a/t} \) and \( d(F^{1-(a/t)} \Delta^{-(m+1)}) \) is an exact differential, and thus has residue zero:

\[
\text{Res} \left( \frac{(1 - \frac{a}{t}) \Delta dF - (m + 1) F d\Delta}{F_0^m \Delta^{m+2}} \right) = 0.
\]

Eliminating the \( F d\Delta \) term and writing \( dF = F'(q) dq \), we obtain our Lemma. \( \diamondsuit \)

Now in each of the cases where we want to find \( N_{m+1} \), the power series \( F_0 \) is an enumerator and thus has nonnegative coefficients, while \( 1/\Delta \) has an expansion \( q^{-1} + \sum_{k=0}^{\infty} c_k q^k \) in which we prove that every coefficient \( c_k \) is positive by writing \( 1/\Delta \) as a product of geometric series. Since \( a \leq t - 1 \), it follows that \( F_0 q^{t-a}/\Delta^{m+1} \) has positive \( q^k \) coefficient for all \( k \geq -(m + 1) \); in particular the \( q^{-1} \) coefficient is positive, so \( N_{m+1} > 0 \).

\(^1\)Over a field \( K \) of characteristic zero we have an exact sequence

\[
0 \to K \to K((q)) \to \Omega^1 K((q)) \to K \to 0,
\]

where the map \( K((q)) \to \Omega^1 K((q)) \) is the differential and the map \( \Omega^1 K((q)) \to K \) is the residue. A change of variable \( z = z_1 q + O(q^2) \) with \( z_1 \in K^* \) preserves everything except possibly the residue map, but then that map must be preserved up to some nonzero scalar, and since \( dz/z \) has the same residue as \( dq/q \) that scalar must be 1.