Comments on Problem Set 9

Math 250a

November 28, 2001

Problem 2. i) There are two cases. Either $c := -N(x)$ is a perfect square in k, or it is not a perfect square.

Everybody got the second case right (use PS7 #1 (ii), and Skolem–Noether). The first case caused more difficulty. It does occur—for example, take

$$A = M_2(k), \quad x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $N(x) = -1$ and $c = 1$ is a perfect square in k. To prove it in this case, let $a \in k$ with $a^2 = c$. Since $x^2 - a^2 = (x-a)(x+a) = 0$, the ring A has zero divisors, so it is not a division ring. Accordingly, it *must* be a matrix algebra. The characteristic polynomial of x is then $X^2 - \mathrm{Tr}(x)X + \det(x) = X^2 - a^2$, so x is a diagonalizable matrix with eigenvalues $\pm a$. The same argument applies to y. Therefore x and y are conjugate, since they diagonalize to the same matrix.

iv) As noted in Prof. Elkies's email, the statement can be false when A is a matrix algebra. The only other possibility is that A is a division ring, and in this case the statement is actually very easy! If $d = 0$, then $N(\mathbf{j}) = -\mathbf{j}^2 = 0$, so then $\mathbf{j}$ is a zero divisor, but $\mathbf{j} \neq 0$, contradicting the assumption that A is a division ring.

Problem 3. $\mathbf{H}_2$ is constructed by choosing $c = d = 1$. By problem 2 (v), one has to show that $r^2 + s^2 + t^2 = 0$ has no solutions in $k = \mathbf{Q}_2$ other than $(0,0,0)$. Suppose one has a nonzero solution. By clearing denominators, one may assume $r, s, t \in \mathbf{Z}_2$. By dividing out extraneous powers of 2, one may assume that one of r, s, t is odd. But $r^2 + s^2 + t^2 \equiv 0 \pmod 4$ doesn't even have any solutions unless r, s, t are all even, so this is a contradiction.