Comments on Problem Set 8

Math 250a

November 14, 2001

Problem 1. For part (i), some students assumed for this problem that A is finite dimensional over k, but this hypothesis is not present in the problem.

Problem 3. In part (ii), some students asserted that A is isomorphic (as an algebra) to a matrix algebra $M_n(k)$ for some n. This is **false** unless the class of A in the Brauer group $\mathrm{Br}(k)$ is trivial. For example, the quaternions $\mathbf{H}$ are not isomorphic to any matrix algebra $M_n(\mathbf{R})$ over $\mathbf{R}$.

In general, the following is true:

- A is isomorphic to $M_n(D)$ for some natural number n and some skew field D containing k.
- $A \otimes_k L$ is isomorphic to $M_n(L)$ for some natural number n and some field extension L of k of finite degree. Such a field L is called a *decomposition field* for A.

To do part (ii) of this problem, take a decomposition field for A, so that $A \otimes_k L \cong M_2(L)$. By the definition of reduced trace, $\overline{x} \otimes 1 \in A \otimes_k L$, with $\overline{x}$ taken in A, equals $\overline{x \otimes 1}$ taken in $M_2(L)$, so applying part (i) of the problem yields the desired identities.

Problem 4. For part (iv), the norm map $N : \mathbf{F}_{q^n} \longrightarrow \mathbf{F}_q$ is a polynomial f of degree n satisfying the requirements. Explicitly, f may be described as follows. Choose a basis $\{v_1, \ldots, v_n\}$ for $\mathbf{F}_{q^n}/\mathbf{F}_q$. Then, for any $(x_1, \ldots, x_n) \in \mathbf{F}_q^n$,

$$f(x_1, \ldots, x_n) := N(x_1 v_1 + \cdots + x_n v_n) = \prod_{i=0}^{n-1} \phi^i(x_1 v_1 + \cdots + x_n v_n) = \prod_{i=0}^{n-1} (x_1 \phi^i(v_1) + \cdots + x_n \phi^i(v_n)),$$

where $\phi(x) := x^q$ is the Frobenius map.