Comments on Problem Set 7

Math 250a

November 5, 2001

Problem 4. Given $\phi \in V^*$ and $a \in K$, the correct definition of $a\phi$ is:

$$(a\phi)(v) := \phi(v)a.$$

The map $a\phi$ must be defined in this way because elements ϕ of V^* must satisfy the property $\phi(bv) = b(\phi(v))$ for all $b \in K$ and $v \in V$. Since $a\phi$ is purportedly an element of V^*, we need $(a\phi)(bv) = b((a\phi)(v))$. Fortunately, this is the case:

$$(a\phi)(bv) = \phi(bv)a = b(\phi(v)a) = b((a\phi)(v)).$$

Some people tried the *incorrect* definition $(a\phi)(v) := \phi(av) = a(\phi(v))$. The problem with this definition is that it does not make $a\phi$ an element of V^*:

$$(a\phi)(bv) = a(\phi(bv)) = a(b(\phi(v))) \neq b((a\phi)(v)) = b(a(\phi(v))).$$

Problem 5. To show that every ideal $I \subset A$ is the annihilator of some $W \subset V$, the first thing to do is to show that I is principal—that is, there exists $a \in A$ such that $I = Aa$.

Among all elements of I, choose $a \in I$ such that $\ker(a)$ has minimal dimension. Since I is an ideal, $Aa \subset I$. We claim that $I \subset Aa$. It suffices to prove that $\ker(a) \subset \ker(b)$ for any $b \in I$, since this implies any endomorphism in I factors through a. Suppose not. Choose $b \in I$ and $v \in \ker(a)$ such that $b(v) \neq 0$. Our hypothesis implies that $\ker(a)$ is not 0, so $\operatorname{Im}(a)$ is not V. Choose $w \in V$ with $w \notin \operatorname{Im}(a)$. Construct a K–linear map $f : V \longrightarrow V$ with the following properties:

- $\operatorname{Im}(f) \subset Kw$
- $f(b(v)) \neq 0$

We claim that $\ker(fb + a) \subsetneq \ker(a)$. If $(fb + a)(x) = 0$, then $a(x) = -fb(x) \subset Kw$, but $Kw \cap \operatorname{Im}(a) = 0$, so $a(x) = fb(x) = 0$ and $x \in \ker(a)$. The two are not equal because $a(v) = 0$ but $(fb + a)(v) = fb(v) \neq 0$. Finally, $fb + a \in I$ since $a, b \in I$, so we have produced an element of I whose kernel has smaller dimension than $\ker(a)$, which is a contradiction.

Now we finish the proof. Let $W = \bigcap_{b \in I} \ker(b)$. The above proof shows that $\ker(a) \subset \ker(b)$ for all $b \in I$. On the other hand, $\ker(a)$ is itself a term in the intersection. Therefore $W = \ker(a)$. Furthermore, $\operatorname{Ann}(W) \subset Aa$ since any endomorphism in A which annihilates W factors through a, and the reverse inclusion $Aa \subset \operatorname{Ann}(W)$ is obvious. It follows that $I = Aa = \operatorname{Ann}(W)$.