Comments on Problem Set 4

Math 250a

October 15, 2001

In general, people did very well on what I thought was a difficult problem set. Good job everyone!

Problem 1. Some people tried to argue that, if a map $f : G \times G \longrightarrow G$ (in this case, the product map) is continuous in each coordinate, then f is continuous. But this is not true in general—the standard counterexample comes from multivariable calculus. Let $f : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Then f is continuous in each coordinate (for any fixed x_0, the function $g(y) := f(x_0, y)$ is continuous, and likewise for any fixed y_0). However, the function f is not itself continuous at $(0,0)$.

The function f defined above is not a group operation. The assertion in question might still hold under more stringent hypotheses which apply to Problem 1 (and I'd be interested in seeing any proofs along these lines), but in any case, it is still sufficiently non–obvious that it should not be asserted without proof.

Problem 4. As Prof. Elkies indicated in his email, Zorn's lemma is not needed in Problem 3. However, it is needed in Problem 4, where you need to use multiple times the fact that any F–automorphism of K extends to an F–automorphism of L, for any intermediate subfield K with $F \subset K \subset L$.

Here is a proper proof of this fact. Let $\sigma : K \longrightarrow K$ be an automorphism of K which fixes F. Consider the set S of all pairs (E, σ_E) where $E \subset L$ is an extension of K and $\sigma_E : E \longrightarrow E$ is an F–automorphism of E that agrees with σ on K. Place a partial ordering on S by defining $(E, \sigma_E) < (E', \sigma_{E'})$ if $E \subset E'$ and σ'_E restricts to σ_E on E. Then

- S is nonempty, because it contains (K, σ).
- Every chain $\{(E_i, \sigma_{E_i})\}$ in S has an upper bound, namely (E, σ_E) where $E = \bigcup_i E_i$ and $\sigma_E(x) := \sigma_{E_i}(x)$ for some choice of i with $x \in E_i$.

By Zorn's Lemma, S contains a maximal element (M, σ_M). We claim $M = L$. If not, choose $x \in L \setminus M$ and let $M' \subset L$ be the splitting field over M of the minimal polynomial of x over F. Then $[M' : M]$ is finite, and we know for finite extensions that the map σ_M extends to an automorphism $\sigma_{M'} : M' \longrightarrow M'$ of the splitting field. Therefore $(M', \sigma_{M'}) > (M, \sigma_M)$, contradicting maximality of (M, σ_M).

Problem 5. The hard part of this problem was showing that $[E : F] < \infty$. Most people tried to show that E has only finitely many embeddings into L, and then derive finiteness of degree from that. It is possible to do it this way, but you have to be careful about taking normal closures when you have to. In particular, a number of people falsely asserted that $[E : F] \leq |\operatorname{Aut}(E/F)|$. This is **not** true when E/F is not normal (for example, take $L = \mathbf{Q}(\sqrt[3]{2}, e^{2\pi i/3})$, $E = \mathbf{Q}(\sqrt[3]{2})$, $F = \mathbf{Q}$).

The easy way to show $[E : F] < \infty$ is to *use the Galois correspondence!* Let H be a closed subgroup of finite index in G. Take $E = L^H$. Any intermediate fields K with $F \subset K \subset E$ correspond bijectively to intermediate subgroups H' with $H \subset H' \subset G$. Since $[G : H]$ is finite, there are only finitely many such subgroups H'. Hence, by the Galois correspondence, there are only finitely many intermediate fields K. This implies $[E : F]$ is finite.