Comments on Problem Set 3

Math 250b

March 20, 2002

Exercise 8.27. For part (b), it is reasonable to assume that A is finite dimensional, since in order to form the Lie algebra of G, one needs that G is a Lie group, and we have only discussed finite dimensional Lie groups (or, for that matter, finite dimensional differentiable manifolds) in this class.

For any path $\gamma : \mathbb{R} \to G$ with $\gamma_0 = e$, differentiating the equation $\gamma_t(ab) = \gamma_t(a)\gamma_t(b)$ using the product rule yields $\gamma_0'(ab) = \gamma_0(a)\gamma_0'(b) + \gamma_0'(a)\gamma_0(b) = a\gamma_0'(b) + \gamma_0'(a)b$ which shows that $\gamma_0' \in T_eG$ is a derivation.

The converse statement, that every derivation is an element of T_eG, requires a separate proof. It does not follow automatically from what was done above. In the book, the various Lie algebra calculations on pp. 112–113 all need to be finalized by a dimension count in order to conclude that the "differentiated" condition fully describes the respective Lie algebra. The case of $\mathfrak{sl}_n\mathbb{R}$ (top of p. 112) is the only one where this dimension count is mentioned explicitly.

A dimension count might be possible for Exercise 8.27 (b), but is complicated by the fact that the dimension of G varies depending on what the multiplication operation on A is. An alternative approach is to observe that, for any $D \in \mathrm{Der}(A)$, we have

$$\begin{aligned}
D(ab) &= aD(b) + D(a)b \\
D^2(ab) &= aD^2(b) + 2D(a)D(b) + D^2(a)b \\
&\vdots \\
D^n(ab) &= \sum_{k=0}^{n} \binom{n}{k} D^k(a)D^{n-k}(b) = n! \sum_{k=0}^{n} \frac{D^k(a)}{k!}\frac{D^{n-k}(b)}{(n-k)!}
\end{aligned}$$

This means

$$\exp(tD)(ab) = \sum_{n=0}^{\infty} \frac{D^n(ab)}{n!} t^n = \left(\sum_{n=0}^{\infty} \frac{D^n(a)}{n!} t^n\right)\left(\sum_{n=0}^{\infty} \frac{D^n(b)}{n!} t^n\right) = \exp(tD)(a)\exp(tD)(b),$$

so the path $t \mapsto \exp(tD)$ has image contained inside G, and has tangent vector at the identity equal to D, which shows that $D \in T_eG$.

Exercise 9.1. Let H be the subgroup of G generated by exponentiating $Z(\mathfrak{g})$, and let H' be the connected component of the identity in $Z(G)$. One has to show that $H \subset H'$ and $H' \subset H$.

$H \subset H'$: Since H is connected, it suffices to show that, for any $X \in Z(\mathfrak{g})$, $\exp X$ commutes with G. Take a small neighborhood $U \subset \mathfrak{g}$ of the identity on which exp is invertible, log is defined, and the Campbell–Hausdorff formula holds. Then, for any $X \in U \cap Z(\mathfrak{g})$ and $Y \in U$,

$$\exp X \exp Y = \exp(X * Y) = \exp(X + Y) = \exp(Y + X) = \exp Y \exp X.$$

Therefore $\exp X$ commutes with everything in $\exp(U)$, which is an open neighborhood of the identity in G, and thus generates G, so $\exp X$ commutes with G. This holds for all $X \in U \cap Z(\mathfrak{g})$; but, since $U \cap Z(\mathfrak{g})$ generates $Z(\mathfrak{g})$, it then holds for all $X \in Z(\mathfrak{g})$.

$H' \subset H$: Since $H' \subset Z(G)$, the map $\Psi_{h'} : G \to G$ is the identity map for $h' \in H'$, so $\mathrm{Ad}(h') = \mathrm{Id}$ for every $h' \in H'$. Differentiating again, we get $\mathrm{ad}(X) = 0 \in \mathrm{End}(\mathfrak{g})$ for all X in the Lie algebra $\mathfrak{h}'$ of H' (since Ad was constant). Hence $\mathrm{ad}(X)(Y) = 0$ for every $X \in \mathfrak{h}'$ and $Y \in \mathfrak{g}$. This shows that $\mathfrak{h}' \subset Z(\mathfrak{g})$, which means that H' (the subgroup generated by exponentiating $\mathfrak{h}'$) is contained in H (the subgroup generated by exponentiating $Z(\mathfrak{g})$).