Comments on Problem Set 2

Math 250b

March 6, 2002

Here are the dimensions of the various Lie groups of Exercise 7.8 (problem 5 on this problem set). The groups $\mathrm{GL}_n \mathbb{R}$, $\mathrm{SL}_n \mathbb{R}$, B_n, N_n, $\mathrm{SO}_n \mathbb{R}$, and $\mathrm{Sp}_{2n} \mathbb{R}$ were done in class, but we give them here because some people even got the ones that were done in class wrong. For each group, you can either directly find the dimension of the Lie group, or find the dimension of the corresponding Lie algebra; the two are the same since a manifold has the same dimension as its tangent space at any point.

$\mathrm{GL}_n \mathbb{R}$ is an open subset of $\mathrm{Mat}_{n\times n}(\mathbb{R})$, so it has dimension n^2. Its Lie algebra is $\mathrm{End}(\mathbb{R}^n) = \mathrm{Mat}_{n\times n}(\mathbb{R})$, which also has dimension n^2.

$\mathrm{SL}_n \mathbb{R}$ is obtained from $\mathrm{GL}_n \mathbb{R}$ by imposing the condition $\det = 1$, which subtracts one degree of freedom. So $\dim \mathrm{SL}_n \mathbb{R} = n^2 - 1$. Its Lie algebra $\mathfrak{sl}_n\mathbb{R}$ is the set of trace 0 matrices, also of dimension $n^2 - 1$.

B_n, the group of invertible upper triangular matrices, is an open subset of the set of all upper triangular matrices, which has dimension $\binom{n+1}{2}$. (The Lie algebra $\mathfrak{b}_n$ is equal to the set of all upper triangular matrices.)

N_n (respectively, $\mathfrak{n}_n$) is like B_n ($\mathfrak{b}_n$) except that the diagonal is fixed with 1's (0's), so the dimension is $\binom{n}{2}$.

$\mathrm{SO}_n \mathbb{R}$ consists of matrices whose columns form an orthonormal basis for $\mathbb{R}^n$. To find its dimension we compute the degrees of freedom for each column. The first column v_1 must satisfy the relation $(\mathbf{v}_1, \mathbf{v}_1) = 1$, which takes away 1 of its n degrees of freedom, so it has $n - 1$ degrees of freedom. The second column must satisfy $(\mathbf{v}_2, \mathbf{v}_2) = 1$ and $(\mathbf{v}_1, \mathbf{v}_2) = 0$, so it has $n - 2$ degrees of freedom. Continuing, we see the dimension is

$$(n-1) + (n-2) + \cdots + 2 + 1 + 0 = \binom{n}{2}.$$

Alternatively, the Lie algebra $\mathfrak{so}_n\mathbb{R}$ consists of $n \times n$ skew–symmetric matrices, so one is free to choose the $\binom{n}{2}$ strictly upper triangular entries arbitrarily but then the rest of the matrix is determined.

$\mathrm{SO}_{k,l} \mathbb{R}$: The same argument that works for $\mathrm{SO}_n \mathbb{R}$ above works for an indefinite form, so $\dim \mathrm{SO}_{k,l} \mathbb{R} = \dim \mathrm{SO}_n \mathbb{R} = \binom{n}{2}$. If you want to use Lie algebras, let $M = \left(\begin{smallmatrix} I_k & 0 \\ 0 & -I_l \end{smallmatrix}\right)$ be the matrix of the bilinear form; then the Lie algebra consists of all X satisfying

$${}^tX \cdot M + M \cdot X = 0$$

(equation 8.21 in the book), and there are $\binom{k}{2} + \binom{l}{2} + kl$ degrees of freedom for constructing such X.

$\mathrm{Sp}_{2n} \mathbb{R}$: An argument similar to that of $\mathrm{SO}_n \mathbb{R}$ above works here, with the following difference: the requirement $(\mathbf{v}_1, \mathbf{v}_1) = 0$ for the first vector is always automatically satisfied, so the dimension in this case is

$$n + (n-1) + \cdots + 2 + 1 = \binom{n+1}{2}.$$

The Lie algebra is $\mathfrak{sp}_{2n}\mathbb{R} = \mathrm{Sym}^2 V$ (equation 8.23 in the book), which also has dimension $\binom{n+1}{2}$.

$U(n)$ can be done by the same argument as $\mathrm{SO}_n\,\mathbb{R}$. The first vector $\mathbf{v}_1$ can be chosen from n complex dimensions (that is, $2n$ real dimensions); however the requirement $(\mathbf{v}_1, \mathbf{v}_1) = 1$ reduces the space of choices by 1 dimension. Similarly, $\mathbf{v}_2$ has $2n-3$ choices of dimension, etc., so $\dim U(n) = (2n-1) + (2n-3) + \cdots + 3 + 1 = n^2$.

The Lie algebra $\mathfrak{u}_n$ is the set of skew–Hermitian matrices, which must be purely imaginary on the diagonal and can be arbitrary above the diagonal, for $n + 2\binom{n}{2} = n^2$ dimensions.

$SU(n)$ has one less dimension than $U(n)$: the determinant of anything in $U(n)$ falls on the unit circle of complex norm 1; requiring the determinant to equal 1 takes a point out of that circle. The Lie algebra $\mathfrak{su}(n)$ is the set of skew–Hermitian matrices of trace 0, which also has dimension n^2-1: a skew–Hermitian matrix must have purely imaginary trace, so setting trace $= 0$ only cuts down one dimension.

$\mathrm{GL}_n\,\mathbb{C}$ (respectively, $\mathfrak{gl}_n\mathbb{C}$), in analogy to $\mathrm{GL}_n\,\mathbb{R}$ ($\mathfrak{gl}_n\mathbb{R}$), has n^2 complex dimensions, so it has $2n^2$ real dimensions.

$\mathrm{SL}_n\,\mathbb{C}$ (respectively, $\mathfrak{sl}_n\mathbb{C}$), in analogy to $\mathrm{SL}_n\,\mathbb{R}$ ($\mathfrak{sl}_n\mathbb{R}$), has n^2-1 complex dimensions, or $2(n^2-1)$ real dimensions.

$\mathrm{GL}_n\,\mathbb{H}$ is an open subset of the space of $\mathbb{H}$–linear endomorphisms of $\mathbb{H}^n$, so it has dimension $4n^2$ over $\mathbb{R}$.

$Sp(n) = U_{\mathbb{H}}(n)$ can be done the same way as $U(n)$: the first vector has $4n-1$ choices; the second $4n-5$; etc., so $\dim Sp(n) = (4n-1) + (4n-5) + \cdots + 3 = 2n^2 + n$.

Alternatively, the Lie algebra $\mathfrak{sp}(n)$ is $\mathfrak{u}(2n) \cap \mathfrak{sp}_{2n}\mathbb{C}$, which consists of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ which are both skew–Hermitian and satisfy

$$\begin{pmatrix} -{}^tC & {}^tA \\ -{}^tD & {}^tB \end{pmatrix} + \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = 0$$

(see p. 239 in the book), which gives n^2 dimensions of choices for A and $2\binom{n+1}{2}$ for B, so $\dim \mathfrak{sp}(n) = n^2 + 2\binom{n+1}{2} = 2n^2 + n$.