Comments on Problem Set 1

Math 250a

September 24, 2001

1. In order to show that $\phi : K \longrightarrow \mathrm{End}_F(K)$, $\phi(a) := M_a$ is a ring homomorphism, you have to check three things:

   - $\phi(a+b) = \phi(a) + \phi(b)$
   - $\phi(ab) = \phi(a)\phi(b)$
   - $\phi(1) = 1$

   The last condition is necessary because Elkies specified in class that for this course all ring homomorphisms must map the identity to the identity. The last condition does **not** follow from the other two conditions, because, as mentioned in class, the map $\mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Z}/6\mathbf{Z}$ given by $1 \mapsto 3$ satisfies the first two conditions and not the third.

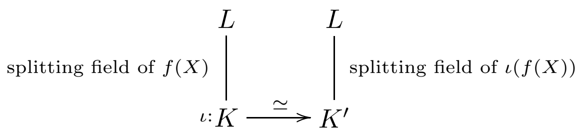
4. A lot of people did not have a correct working definition of $F(u)$ for this problem. In this context, $L := F(u)$ is defined to be the field of (abstract) rational functions over F in one indeterminate u. So

   $$F(u) = \text{quotient field of } F[u] = \left\{ \frac{p(u)}{q(u)} \;\middle|\; p(u), q(u) \in F[u] \text{ and } q(u) \neq 0 \right\} \text{ modulo the usual relations.}$$

   In particular, there are elements of $F(u)$ that are not polynomials, and there are elements of $F(u)$ that are not finite Laurent series $\sum_{k=-n}^{k=n} a_k u^k$ for some n.

6. Most people tried induction on $[L : K]$. While this method can be made to work (and many people succeeded at it), a much easier approach is to leverage Jacobson Theorem 4.4 which was already proved in class. We give this proof here.

   L is a splitting field of $f(X)$ over F, and thus *a fortiori* is a splitting field of $f(X)$ over K (since K contains F). Let K' be the image of ι in L. By abstract algebra, $\iota : K \longrightarrow K'$ is an isomorphism of fields. Since K' contains F, the field L is also a splitting field of $f(X)$ over K'. Finally, since $\iota(f(X)) = f(X)$, we conclude that L is a splitting field of $\iota(f(X))$ over K'.

   By Jacobson Theorem 4.4, the isomorphism $\iota : K \longrightarrow K'$ extends to an isomorphism $L \longrightarrow L$ of the splitting fields.