Math 250a: Higher Algebra
Problem Set #9 (9 November 2001):
Baer multiplication and quaternion algebras

Let G be any group acting on a abelian group A. Having already identified $H^2(G, A)$ with the isomorphism classes of extensions $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$, we gave Baer's recipe for starting from two such extensions E, E' and obtaining the extension $E'' = (E, E')/Q$ that corresponds to their sum in $H^2(G, A)$. We next obtain further information about this operation that does not depend on the cohomological interpretation: we identify the identity element and inverse, and show commutativity and associativity.

1. **Solve any two of the following three parts.**
   i) Show that if E is the semidirect product $A \rtimes G$ then $E'' \cong E'$. [Thus the semidirect product is the identity element for Baer multiplication. Recall that $E \cong A \rtimes G$ if and only E has a "section", that is, a homomorphism $s : G \to E$ such that $\pi(s(g)) = g$ for all $g \in G$.]
   ii) Prove that Baer multiplication is commutative by showing that $(E, E')/Q$ and $(E', E)/Q$ are isomorphic extensions of G by A. If E_1, E_2, E_3 are extensions of G by A with the same G-action on A, let
   $$(E_1, E_2, E_3) := \{(e_1, e_2, e_3) \in E_1 \times E_2 \times E_3 : \pi_1(e_1) = \pi_2(e_2) = \pi_3(e_3)\}$$
   and $Q_2 := \{(a_1, a_2, a_3) \in A^3 : a_1 a_2 a_3 = 1\}$. Prove that (E_1, E_2, E_3) is an extension of G by A^3, that Q_2 is a normal subgroup of (E_1, E_2, E_3), and that $(E_1, E_2, E_3)/Q_2$ is an extension of G by A with G acting on A as it does for each E_i. Show that Baer multiplication is associative by identifying $(E_1, E_2, E_3)/Q_2$ with the Baer products of E_1, E_2, E_3 taken in either order.
   iii) For any extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$, let E^o be the extension $1 \to A \xrightarrow{-\iota} E \xrightarrow{\pi} G \to 1$ with the opposite embedding of A in E. [Why do E, E^o have the same G-action on A?] Prove that E^o is the inverse of E in two ways: by identifying $(E, E^o)/Q$ with the semidirect product $A \rtimes G$, and by showing that E, E^o correspond to inverse elements of $H^2(G, A)$.

Note that the formula for E^o is what one might expect from the special case $(G, A) = (\mathrm{Gal}(L/k), L^*)$ and our results about the opposite of a central simple algebra.

We next describe generalized quaternions over an arbitrary field k not of characteristic 2. Let A be a central simple k-algebra of dimension 4. In the previous problem set we showed that A has an anti-involution $x \leftrightarrow \bar{x} = \mathrm{Tr}(x) - x$, with $\mathrm{Tr} : A \to k$ being the reduced trace. Let A_0 be the kernel of Tr; it is a k-vector subspace of A of dimension $4 - 1 = 3$. Let $N : A \to k$ be the reduced norm, so $N(x) = x\bar{x}$. This is a quadratic form on A, and the associated bilinear form is

$$(x, y) = N(x + y) - N(x) - N(y) = x\bar{y} + y\bar{x} = \mathrm{Tr}(x\bar{y}).$$

Note that if $x \in A_0$ then $N(x) = -x^2$.

2. i) Prove that if $x, y \in A_0$ with $N(x) = N(y) \neq 0$ then x, y are conjugate in A. [Hint: see PS7 #1(ii).]

   ii) Prove that there exist $\mathbf{i} \in A_0$ with $N(\mathbf{i})$ nonzero. Fix one such $\mathbf{i}$, and let $c = N(\mathbf{i})$. Since also $N(-\mathbf{i}) = N(\mathbf{i})$, by part (i) there exist invertible $z \in A$ such that $\mathbf{i}z = -z\mathbf{i}$. Show that $\mathbf{i}\bar{z} = -\bar{z}\mathbf{i}$, and hence that $\mathbf{ij} = -\mathbf{ji}$ where $\mathbf{j} := z - \bar{z}$. Show that $\mathbf{j} \in A_0$ and $\mathbf{j} \neq 0$.

   iii) Now let $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$. Show that $\mathbf{k} \in A_0$ and $\mathbf{ki} = -\mathbf{ik} = c\mathbf{j}$. Let $d = N(\mathbf{j}) = -\mathbf{j}^2$, and determine $\mathbf{jk}, \mathbf{kj}, \mathbf{k}^2$ in terms of $c, d, \mathbf{i}, \mathbf{j}, \mathbf{k}$. In particular show that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pairwise orthogonal for the bilinear form $(\cdot, \cdot)$.

   iv) Prove that $d \neq 0$. (Assume that $d = 0$ and construct a nontrivial two-sided ideal in A.) Conclude that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are linearly independent, and thus that $A = k + k\mathbf{i} + k\mathbf{j} + k\mathbf{k}$. Since we know the multiplication table of $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, this determines A.

   v) Prove that A is a division ring if and only if there are no $(r, s, t) \in k^3$ such that $cr^2 + ds^2 + cdt^2 = 0$ other than $(0, 0, 0)$.

It can be shown that every nondegenerate quadratic form on k^3 is equivalent to a multiple of $cr^2 + ds^2 + cdt^2 = 0$ for some $c, d \in k^*$; these c, d are <u>not</u> uniquely determined by the form, but the central simple algebras A associated to the quadratic form <u>is</u> uniquely determined by the equivalence class of the quadratic form up to scaling, and vice versa. Starting from part (i) we can also identify $A^*/\{\pm 1\}$ with the group of k-linear transformations of A_0 of determinant 1 that preserve the bilinear form $(\cdot, \cdot)$. This generalizes the identification of $\mathbf{H}^*/\{\pm 1\}$ with $\mathrm{SO}_3(\mathbf{R})$. If we regard $cr^2 + ds^2 + cdt^2 = 0$ as a conic in the projective plane over k, we get the simplest example of a "Brauer-Severi variety" associated to a central simple algebra.

If $k = \mathbf{R}$ and A is a division ring, then clearly $c, d > 0$; we may then scale $\mathbf{i}, \mathbf{j}$ by $c^{1/2}, d^{1/2}$ to identify A with $\mathbf{H}$. This completes the proof that $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ are the only division algebras of finite dimension over $\mathbf{R}$. Likewise it can be shown that for each p there is a unique division algebra $\mathbf{H}_p$ with center $\mathbf{Q}_p$ and of dimension 4 over $\mathbf{Q}_p$. For odd p we constructed $\mathbf{H}_p$ in the last problem set. For our final problem, we treat the even case:

3. Construct a division ring $\mathbf{H}_2$ with center $\mathbf{Q}_2$ and of dimension 4 over $\mathbf{Q}_2$.

You won't have to look very long for suitable c, d!

Problems 1–3 are due in class Monday, November the 19th.

4. Send me e-mail, or schedule a time to meet with me, to discuss your final paper topic.