Math 250a: Higher Algebra
Problem Set #8 (2 November 2001):
Tensor products, quaternion algebras, and Chevalley

1. i) Let A be any algebra over a field k. Prove that $A \otimes_k M_n(k)$ is isomorphic as a k-algebra with $M_n(A)$, for each positive integer n.
   ii) Prove that the tensor product $M_m(k) \otimes_k M_n(k)$ is isomorphic as a k-algebra with $M_{mn}(k)$ for all positive integers m, n (and an arbitrary commutative field k).

2. Find 2×2 matrices I, J, K with entries in $\mathbf{C}$ such that

$$I^2 = J^2 = K^2 = -\mathbf{1}, \quad IJ = -JI = K, \ JK = -KJ = I, \ KI = -IK = J.$$

   Use these to construct an explicit isomorphism from $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}$ to $M_2(\mathbf{C})$, and use the isomorphism to verify that the reduced trace and reduced norm of any $x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{H}$ are $2a$ and $a^2 + b^2 + c^2 + d^2$ respectivly.

3. i) For any (commutative) field k, define a map $x \mapsto \bar{x}$ on $M_2(k)$ by $\bar{x} = \mathrm{Tr}(x) \cdot \mathbf{1} - x$. Here $\mathrm{Tr}(x)$ is the trace of x as a 2×2 matrix, and $\mathbf{1}$ is the 2×2 identity matrix, which is the unit element of $M_2(k)$. Prove that this map is an anti-involution, i.e., that it satisfies the identities $\bar{\bar{x}} = x$ and $\overline{xy} = \bar{y}\bar{x}$. [This can be done either by explicit computation or via a relation between $\bar{x}$ and the transpose of x.]
   ii) Now suppose that A/k is any central simple algebra with $\dim_k A = 4$. Define a map $x \mapsto \bar{x}$ on A by $\bar{x} = \mathrm{Tr}(x) \cdot 1 - x$, where $\mathrm{Tr}(x)$ is the reduced trace of x and 1 is the unit element of A. Prove that this map is an anti-involution.

Next week, we'll use this anti-involution to describe 4-dimensional skew fields over k, and in particular to show that when $k = \mathbf{R}$ the only such skew field is $\mathbf{H}$. For now, we note that since A is thus isomorphic with A^o, the class of A in $\mathrm{Br}(k)$ is a 2-torsion element.

We next obtain Chevalley's theorem on solutions of polynomial equations in many variables over a finite field, and deduce the triviality of $\mathrm{Br}(k)$.

Fix a finite field $k = \mathbf{F}_q$, and let p be its characteristic, so q is some power of p.

4. i) Prove that $\sum_{x \in k} x^m = 0$ for all nonnegative integers $m < q - 1$. What is $\sum_{x \in k} x^m$ for an arbitrary nonnegative $m \in \mathbf{Z}$?

It follows that $\sum_{x \in k} P(x) = 0$ for any polynomial $P \in k[X]$ of degree less than $q - 1$. We next extend this to polynomials in several variables $X_1, \ldots, X_n$. The "degree" of a nonzero monomial $c \prod_{i=1}^n X_i^{m_i}$ is $\sum_i m_i$; the degree of a sum of distinct monomials is the largest of those monomials' degrees. This defines the degree on $k[X_1, \ldots, X_n]$. (Note that this degree is invariant under an invertible linear change of variables; thus we may speak of a polynomial of degree m on an n-dimensional vector space over k without specifying which coordinates on that space we use.)

ii) Prove that

$$\sum_{(x_1,\ldots,x_n)\in k^n} P(x_1, \ldots, x_n) = 0$$

for every polynomial $P \in k[X_1, \ldots, X_n]$ of degree less than $n(q-1)$.

iii) Now let $f \in k[X_1, \ldots, X_n]$ be a polynomial of degree less than n. Take $P = f^{q-1}$ in (ii) to prove that the number of solutions in k^n of the equation $f(x_1, \ldots, x_n) = 0$ is a multiple of p.

It follows that every finite skew field is commutative. Indeed, let K be such a field, and k its center. Then K is a vector space over k, of dimension n^2 for some positive integer n. The reduced norm is a polynomial of degree n on that vector space that vanishes only at the origin. Hence $n \geq n^2$. Therefore $n = 1$, and $K = k$ as claimed.

iv) Show that the degree bound in (iii) is sharp by constructing, for each $n = 1, 2, 3, \ldots$, a polynomial f of degree n on an n-dimensional k-vector space V such that f vanishes at the origin but at no other point in V.

Can you generalize Chevalley's theorem to simultaneous solutions of several polynomials of low degree? Can you get a formula for the enumeration mod p of solutions of $x^3 + y^3 + z^3 = 0$ and $x^4 + y^4 = z^2$ in k^3, or $x^3 - x = y^2$ in k^2?

Problems 1–4 are due in class Friday, November the 9th.

5. If you have not done so yet, start to think about final paper topics.