Math 250a: Higher Algebra
Problem Set #6 (19 October 2001):
Group and Galois cohomology, continued

Here's an equivalent description of group cohomology that more closely resembles the simplicial (co)homology you may have seen in the context of topology:

1. [Homogeneous cochains] Let G be a group, and A a G-module. To any r-cochain $f : G^r \to A$ associate the function $f_{\text{homog}} : G^{r+1} \to A$ defined by

$$f_{\text{homog}}(a_0, a_1, \ldots, a_r) := a_0 f(a_0^{-1} a_1, a_1^{-1} a_2, \ldots, a_{r-1}^{-1} a_r).$$

Prove that f_{homog} is "homogeneous" in the sense that

$$g\, f_{\text{homog}}(a_0, a_1, \ldots, a_r) := f_{\text{homog}}(g a_0, g a_1, \ldots, g a_r)$$

for all $g, a_i \in G$, and show that every homogeneous $F : G^{r+1} \to A$ is f_{homog} for a unique r-cochain $f : G^r \to A$, namely the one defined by

$$f(x_1, \ldots, x_r) = F(1, x_1, x_1 x_2, \ldots, x_1 x_2 x_3 \ldots x_r).$$

For any $F : G^{r+1} \to A$, define a function $dF : G^{r+2} \to A$ by

$$\begin{aligned}(dF)(a_0, a_1, \ldots, a_{r+1}) \quad := \quad & F(a_1, a_2, \ldots, a_{r+1}) - F(a_0, a_2, a_3, \ldots, a_{r+1}) \\ & + F(a_0, a_1, a_3, \ldots, a_{r+1}) - + \cdots \\ & + (-1)^{r+1} F(a_0, a_1, a_2, \ldots, a_r).\end{aligned}$$

Show that $d^2 = 0$, that is, that $d(dF) = 0$ for any F. Check that if $F : G^{r+1} \to A$ is homogeneous then so is dF. Finally, show that $d(f_{\text{homog}}) = (\delta f)_{\text{homog}}$ for every r-cochain $f : G^r \to A$, and conclude that $\delta^2 = 0$.

Next, we complete the explanation of why the theorem $H^1(K/k, K^*) = 0$ is called "Hilbert Satz 90" even though Hilbert's theorem is equivalent only to the special case of cyclic $\text{Gal}(K/k)$.

2. Solve the first part of Exercise 6.4(i) in the Tate handout [exactness of the beginning

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H}$$

of the inflation-restriction sequence associated to a normal (closed) subgroup H of G].

3. Use this to deduce the vanishing of $H^1(K/k, K^*)$ when $\text{Gal}(K/k)$ is a (finite) p-group from the special case of a cyclic group.

By Exercise 4.2 it follows that $H^1(K/k, K^*)$ is trivial for any finite normal extension K/k, and we have seen already that by compactness of $\text{Gal}(K/k)$ this is enough to prove $H^1(K/k, K^*) = 0$ in general. This shows how, with enough cohomological tools, we can recover the general result "automatically" from Hilbert's special case of a cyclic cover.

Noncommutative group cohomology. Let G be a group acting by automorphisms on another group M. (As usual, if G has a topological structure, we'll ask that this map $G \times M \to M$ be continuous, and will mainly be concerned with the case that G is a Galois group and M has discrete topology.) We may then think of M^G, the subgroup of M fixed by G, as "$H^0(G, M)$"; but, unless M is commutative, we cannot build the full $H^r(G, M)$ edifice. Still we can construct a useful $H^1(G, M)$. While this is not a group, it has a distinguished point (analogous to the origin in the commutative case) and a "long exact sequence" of sorts.

The set $H^1(G, M)$ is still defined as "cocycles" modulo an equivalence classes. A "cocycle" is a (continuous) map $f : G \to M$ such that $f(\sigma\tau) = f(\sigma)\,\sigma f(\tau)$ holds for all $\sigma, \tau \in G$. Two cocycles f, g are said to be equivalent ("cohomologous") if there exists $m \in M$ such that $g(\sigma) = m^{-1} f(\sigma)\sigma(m)$ for all $\sigma \in G$. The distinguished point in $H^1(G, M)$ is the class of the cocycle sending every every element of G to the identity of M.

4. i) Verify that "is cohomologous to" is an equivalence relation, and that if $\sigma \mapsto f(\sigma)$ is a cocycle then so is $\sigma \mapsto m^{-1} f(\sigma)\sigma(m)$ for every $m \in M$.
   ii) Suppose $1 \to M' \to M \to M'' \to 1$ is a short exact sequence of groups with a (continuous) G-action. Verify that the induced sequence $1 \to M'^G \to M^G \to M''^G$ is exact. Construct a map $\delta : M''^G \to H^1(G, M')$ — which should specialize to the connecting homomorphism $\delta : H^0(G, M'') \to H^1(G, M')$ when M is abelian — and show that an element of M''^G is in the image of M^G if and only if δ maps it to the distinguished element of $H^1(G, M')$.
   iii) Show further that the image of δ consists of those elements of $H^1(G, M')$ that map to the distinguished element of $H^1(G, M)$, and that the image of the map $H^1(G, M') \to H^1(G, M)$ consists of those elements of $H^1(G, M)$ that map to the distinguished element of $H^1(G, M'')$.

In particular, if M is commutative then we have checked the exactness of the sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \to H^1(G, M') \to H^1(G, M) \to H^1(G, M'').$$

We next generalize the theorem $H^1(K/k, K^*) = 0$ from $K^* = \mathrm{GL}_1(K)$ to $\mathrm{GL}_n(K)$. That is, we prove that every continuous cocycle $f : \mathrm{Gal}(K/k) \to \mathrm{GL}_n(K)$ is of the form $f(\sigma) = A\sigma(A^{-1})$ for some $A \in \mathrm{GL}_n(K)$.

5. i) Explain why it is enough to prove this when the normal extension K/k is of finite dimension.
   ii) Prove that $f(\sigma) = A\sigma(A^{-1})$ if and only if each of the column vectors v of A satisfies $v = f(\sigma)\sigma(v)$ for every $\sigma \in G$.
   iii) Assume $[K : k] < \infty$. Generalizing our proof of Theorem 4.29, construct for each $w \in K^n$ a vector $v(w) \in K^n$ such that $v(w) = f(\sigma)\sigma(v(w))$ for every $\sigma \in G$.

6. i) Prove that the K-span of $\{v(w) : w \in K^n\}$ is all of K^n. (If not, then some nonzero K-linear functional u sends each $v(w)$ to zero. Apply this condition to aw for each $a \in K$ to reach a contradiction.) This together with Problem 5 completes the proof of $H^1(K/k, \mathrm{GL}_n(K)) = 0$.
   ii) What are $H^1(K/k, M)$ where M is the $ax + b$ group over K, or the group $\mathrm{SL}_n(K)$?

Problem set is due in class Friday, October the 26th.