Math 250b: Higher Algebra

Problem Set #5 (1 April 2002):
Invariant rings and their Hilbert functions, cont'd; Weyl formula for SL_3;
Killing/Cartan stuff

1. Let V, W be two representations of $G = \mathrm{SL}_2$. Then G acts on $\mathbf{C}[V, W]$, a "bi-graded" commutative ring whose (m, n) homogeneous part is $\mathrm{Sym}^m V \otimes \mathrm{Sym}^n W$. This bi-grading is inherited by the subring $(\mathbf{C}[V, W])^G$ of G-invariant polynomials in V, W.[1] The Hilbert series of a bi-graded ring is a generating function of two variables, say $H(y, z) = \sum_{m,n=0}^{\infty} d(m, n) y^m z^n$ where $d(m, n)$ is the dimension of the (m, n) homogeneous part.
   i) Show that the Hilbert series of $\mathbf{C}[V, W]$ is $(1 - y)^{-\dim V}(1 - z)^{-\dim W}$.
   ii) How would you compute the Hilbert series of $(\mathbf{C}[V, W])^G$ from the characters of $\mathrm{diag}(\lambda, \lambda^{-1})$ acting on V, W?
   iii) Carry this computation out in the case that $V = W =$ the three-dimensional adjoint representation of G. Use the resulting formula
$$H(y, z) = \frac{1}{(1 - y^2)(1 - yz)(1 - z^2)}$$
   to determine the ring of G-invariants in this case.

[If you're eager for a more challenging example, you can try the next few cases of irreducible representations V, W. Let V, W have dimensions $d + 1$ and $e + 1$, so that they are Sym^d and Sym^e of the defining two-dimensional representation. Thus part (iii) concerned the case $(d, e) = (2, 2)$. If I compute correctly, $(d, e) = (2, 3)$, $(2, 4)$, $(3, 3)$ yield the following Hilbert functions $H(y, z)$:
$$\frac{1 + y^3 z^4}{(1 - y^2)(1 - z^4)(1 - yz^2)(1 - y^3 z^2)}, \quad \frac{1 + y^3 z^3}{(1 - y^2)(1 - z^2)(1 - z^3)(1 - y^2 z)(1 - y^2 z^2)},$$
$$\frac{(1 + y^2 z^2)(1 + y^3 z^3)}{(1 - y^4)(1 - z^4)(1 - yz)(1 - yz^3)(1 - y^3 z)}.$$
What are the corresponding invariant rings? Likewise for triple products; e.g., for $(d, e, f) = (1, 1, 1)$ and $(2, 2, 2)$ obtain and explain the Hilbert functions $1/(1 - xy)(1 - xz)(1 - yz)$ and
$$\frac{1 + xyz}{(1 - x^2)(1 - y^2)(1 - z^2)(1 - xy)(1 - xz)(1 - yz)}. \quad]$$

[1] NB G is acting "diagonally", that is, simultaneously on V and W; if we required invariance under the action of G on V and W separately, we would be dealing with the invariants of $\mathbf{C}[V, W]$ under an action of $G \times G$, and the result would be simply $(\mathbf{C}[V])^G \otimes (\mathbf{C}[W])^G$.

Let G be a semisimple Lie group with Cartan subgroup H. The *Weyl character formula* (Chapter 24 ff.) gives for each irreducible representation V of G the character of an arbitrary element of H acting on V. We derived this formula for $G = \mathrm{SL}_2$ from our explicit description of the irreducible representations V_n of this group. The next exercise does the same for the irreducible representations $\Gamma_{a,b}$ of SL_3, using "Claim 13.4" (pages 183–5). It may also be regarded as an alternative proof of that Claim assuming that the WCF is already known, as suggested on page 181.
In this case H consists of the diagonal matrices $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1\lambda_2\lambda_3 = 1$. The formula will use quotients of the determinants

$$D(m_1, m_2, m_3) := \begin{vmatrix} \lambda_1^{m_1} & \lambda_2^{m_1} & \lambda_3^{m_1} \\ \lambda_1^{m_2} & \lambda_2^{m_2} & \lambda_3^{m_2} \\ \lambda_1^{m_3} & \lambda_2^{m_3} & \lambda_3^{m_3} \end{vmatrix}.$$

For instance, $D(2,1,0)$ is the Vandermonde determinant $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Each $D(m_1, m_2, m_3)$ is a polynomial in $\lambda_1, \lambda_2, \lambda_3$ divisible by $D(2,1,0)$; the quotient is a special case of the Schur functions that we already used to describe the characters of the symmetric group.

2. i) Prove that $D(m_1, m_2, m_3) = D(m_1 + n, m_2 + n, m_3 + n)$, and that the product $D(m_1, m_2, m_3)D(n_1, n_2, n_3)$ is the determinant of the 3×3 matrix with entries $p(m_i + n_j)$ where $p(k)$ is the power sum $\lambda_1^k + \lambda_2^k + \lambda_3^k$.
   ii) Using (i) or otherwise, show that the character of $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ acting on $\Gamma_{a,b}$ is $D(a + b + 2, b + 1, 0)/D(2, 1, 0)$.

[Hint: for $b = 0$, we are dealing with $\mathrm{Sym}^a V$, which we already dealt with in class (remember the triangle of 1's?); for $a = 0$ we have $\mathrm{Sym}^b V^*$ which can be handled similarly. This gives the character of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, and then we can use Claim 13.4 to finish the computation.]

About the Killing form and Cartan algebra:

3. [Cf. Exercises 14.32 and 14.36* on pages 209 and 210] Show by direct computation that the Killing form on $\mathfrak{gl}_m$ is given by $B(X, Y) = 2m\mathrm{Tr}(XY) - 2\mathrm{Tr}(X)\mathrm{Tr}(Y)$ for $m \times m$ matrices X, Y. Restricting B to $\mathfrak{sl}_m$, verify the statements of §14.2 for $\mathfrak{sl}_m$.

4. Solve Exercise D.5 on page 488.

Problem set is due in class Wednesday the 10th of April.