Math 250b: Higher Algebra

Problem Set #4 (13 March 2002):
More basics about Lie groups and algebras, and some SL_2 "plethysm"

1.–4. Solve problems 9.5, 9.23, 9.25*, and 10.21 in the Fulton-Harris textbook (pages 124, 131, and 145).

For the next two problems, let V be a two-dimensional vector space over $\mathbf{C}$, and let G be the Lie group $\mathrm{SL}(V) \cong \mathrm{SL}_2$. Recall that $\mathrm{Sym}^n(V)$ may be regarded as the space of homogeneous polynomials of degree n in two variables x, y.

5. i) Determine for each $n = 0, 1, 2, \ldots$ a generator B_n for the one-dimensional space of G-maps from $\mathrm{Sym}^n(V) \otimes \mathrm{Sym}^n(V)$ to $\mathbf{C}$. Check that your map B_n is symmetric or alternating according as n is even or odd.
   ii) Now suppose $P \in \mathrm{Sym}^4(V)$ has distinct roots. Prove that $B_4(P \otimes P) = 0$ if and only if these roots have tetrahedral symmetry, i.e., iff P is equivalent with $x^4 - xy^3$ under $\mathrm{GL}(V)$.

6. Show that for any nonnegative integers a, b the Jacobian determinant

$$(P, Q) \mapsto \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}$$

   yields a map from $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$ to $\mathrm{Sym}^{a+b-2}(V)$ *considered as representations of* G, and thus generates the one-dimensional space of G-maps from $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$ to $\mathrm{Sym}^{a+b-2}(V)$. If $a = b$, why does this yield a map from $\wedge^2(\mathrm{Sym}^a(V))$ to $\mathrm{Sym}^{2a-2}(V)$?

Problem set is due in class Friday the 22nd of March.