

Math 250b: Higher Algebra

Problem Set #2 (20-02-2002):

Representations of S_d and other finite groups; a bit about Lie groups

The characters of a finite group G can be used to enumerate solutions of some equations in G . For example:

1. i) Let C be a conjugacy class in a finite group G , and c the element of the group algebra $\mathbf{C}[G]$ defined by $c = \sum_{g \in C} g$. For each irreducible representation V of G , prove that the image of c in $\text{End}(V)$ is scalar multiplication by some $N_V(C)$, and show that $N_V(C) = (|C|/\dim(V)) \chi(C)$ where χ is the character of V .
- ii) Now let C_i ($i = 1, \dots, m$) be some conjugacy classes in G , and c_i the corresponding sums $\sum_{g_i \in C_i} g_i$. Then the constant coefficient of $c_1 c_2 c_3 \cdots c_m$ is the number of solutions of $g_1 g_2 g_3 \cdots g_m = 1$ with $g_i \in C_i$ for each i . Show that this coefficient equals $|G|^{-1} \sum_V \dim^2(V) N_V(C_1) N_V(C_2) N_V(C_3) \cdots N_V(C_m)$.

We illustrate with one application; for others, see for instance the section on the “rigidity method” in Serre’s *Topics in Galois Theory*.

2. Let T be a linear transformation of a vector space V of finite dimension n , with characteristic polynomial $P(z) = \det(zI - T) = z^n + \sum_{i=1}^n (-1)^i c_i z^{n-i}$. Prove that c_i is the trace of the induced action of T on the i -th exterior power $\wedge^i V$, for each $i = 1, 2, \dots, n$. [Hint: start with the case of a diagonalizable operator.]
3. Now let V be the standard representation of S_d (so $n = d - 1$), and T the image of a d -cycle. Use the result of the previous problem to solve Exercise 4.16* (page 51). [For the non-hooks, use Ex. 2.21 (p.17–18), which we did towards the end of 250a.]
4. i) Apply the same result to a $(d-1)$ -cycle to show that its character on $\wedge^i V$ vanishes except for the one-dimensional cases $i = 0$ and $i = d - 1$.
- ii) Combining this with the formulas in Problem 1, show that every odd permutation in S_d ($d > 2$) can be written in $2(d-2)!$ ways as a product of a d -cycle and a $(d-1)$ -cycle.

Next, a couple of introductory problems on Lie groups. Yes, 7.11 is starred; you’re welcome to the hint on page 522. For 7.8, you won’t have to do each of the 13 real Lie groups because we’ll compute several of these dimensions in class.

5. Solve the parts of Exercise 7.8 (page 101) that aren’t done in class by the end of the Feb.27 meeting.
6. Solve Exercise 7.11* on page 101.

Problem set is due in class Friday the 1st of March.