Math 250b: Higher Algebra
Invariants of binary quadrics, etc.

Proposition. *Let T be a linear transformation of a finite-dimensional vector space W, with eigenvalues c_i. Then the character of the induced action of T on $\mathrm{Sym}^n W$ is the coefficient of z^n in the generating function $\prod_i (1 - c_i z)^{-1}$.*

(Cf. PS2 #2; again this is easy when T is diagonalizable, which is the only case we shall use.)

Example: Suppose W is the defining two-dimensional represenation V of SL_2, and $c_1 = \lambda$, $c_2 = \lambda^{-1}$. Then our generating function is

$$\frac{1}{(1-\lambda z)(1-\lambda^{-1}z)} = \frac{1}{\lambda - \lambda^{-1}} \left(\frac{\lambda}{1-\lambda z} - \frac{\lambda^{-1}}{1-\lambda^{-1}z} \right)$$

so the coefficient of z^n is $(\lambda^{n+1} - \lambda^{-(n+1)})/(\lambda - \lambda^{-1})$, which we recognize as the character of T acting on $\mathrm{Sym}^n W$.

In general, if W is a representation of SL_2, we can use this to get at the ring of SL_2-invariant polynomials in W by computing the dimension of the space of invariants of degree n. Recall that for any representation of SL_2 the dimension of the invariant subspace is the constant coefficient of the polynomial obtained by multiplying the character of $\mathrm{diag}(\lambda, \lambda^{-1})$ by $1 - \lambda^2$. For instance, for $\mathrm{Sym}^n V$ we get

$$(1-\lambda^2)\frac{\lambda^{n+1} - \lambda^{-(n+1)}}{\lambda - \lambda^{-1}} = \lambda^{-n} - \lambda^{n+2},$$

confirming that the only invariant polynomials are the constants.

For a slightly more complicated example, let W be the adjoint representation $\mathrm{Sym}^2 V$. Then $c_1, c_2, c_3 = \lambda^2, 1, \lambda^{-2}$; to reduce notational clutter let $u = \lambda^2$. Then the partial-fraction decomposition of our generating function is

$$\frac{1-u}{(1-uz)(1-z)(1-u^{-1}z)} = \frac{u^3}{1-u^2}\frac{1}{1-uz} + \frac{1}{1-u^2}\frac{1}{1-u^{-1}z} - \frac{u}{1-u}\frac{1}{1-z}$$

so the dimension of SL_2 invariants in $\mathrm{Sym}^n W$ is the constant coefficient of

$$\frac{u^{n+3}}{1-u^2} + \frac{u^{-n}}{1-u^2} - \frac{u}{1-u} = \frac{(u^{n+1}-1)(u^2-u^{-n})}{1-u^2}.$$

The first or second factor of the numerator is a multiple of $(1-u^2)$ in $\mathbf{C}[u, u^{-1}]$ according as n is odd or even. Canceling this common factor, we obtain the decomposition of $\mathrm{Sym}^n(W)$ into irreducibles, as asked in Exercise 11.14 (page 153):

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_{\alpha=0}^{[n/2]} \mathrm{Sym}^{2n-4\alpha} V .$$

In particular, the space of SL_2-invariant polynomials of degree n has dimension 1 or 0 according as n is even or odd. We already know for each even n an invariant of degree n, namely the $(n/2)$nd power of the discriminant Δ of a quadratic polynomial. Hence $\mathbf{C}[\Delta]$ is the full ring of invariants. [This is the first instance of a celebrated theorem on the ring of invariant polynomials of a simple Lie group acting on its adjoint representation. Can you describe for each α an SL_2-invariant map from $\mathrm{Sym}^2 V$ to $\mathrm{Sym}^{2n-4\alpha} V$, and thus solve the plethysm problem associated to Ex. 11.14?]

When W is of yet higher dimension, we need a more systematic way to keep track of the constant coefficients of our generating functions. For instance, consider the case $W = \mathrm{Sym}^4 V$. We already know that the space of invariants in $\mathrm{Sym}^2 W$ is $\mathbf{C}\Delta_2$ for some quadratic polynomial whose vanishing detects quartics with tetrahedral symmetry. One likewise computes that the invariants in $\mathrm{Sym}^3 W$ are again one-dimensional, say $\mathbf{C}\Delta_3$ (see Exercise 11.25* on page 158); as it turns out the vanishing of Δ_3 detects quartics like $x^4 - y^4$ with dihedral symmetry. Now Δ_2, Δ_3 are algebraically independent: else they would be proportional to the square and cube of an invariant linear polynomial, which does not exist. We shall show that $\mathbf{C}[\Delta_2, \Delta_3]$ is the full ring of invariants by proving that the dimension of invariants in $\mathrm{Sym}^n W$ is the z^n coefficient of the generating function $1/(1-z^2)(1-z^3)$.

Our method is most easily described in terms of complex analysis of one variable (though it can be framed algebraically too). We want the u^0 coefficient of

$$F(u,z) := \frac{1-u}{(1-u^2z)(1-uz)(1-z)(1-u^{-1}z)(1-u^{-2}z)}$$

as a power series in z. That coefficient is $(2\pi i)^{-1} \oint_{|u|=1} F(u,z)\, du/u$. For small nonzero z, the contour contains three simple poles: at $u = z$ and $u = \pm\sqrt{z}$. The residues are $-z/(1-z)(1-z^2)(1-z^3)$ and $1/2(1-z)(1 \pm z^{3/2})(1-z^2)$. Since $1/(1-z^{3/2}) + 1/(1+z^{3/2}) = 2/(1-z^3)$, the sum of the residues is $(1-z)/(1-z)(1-z^2)(1-z^3)$, which simplifies to $1/(1-z^2)(1-z^3)$ as promised.

Can you show similarly that the ring of invariants of $\mathrm{Sym}^3 V$ is $\mathbf{C}[\Delta_4]$, where Δ_4 is the discriminant of a cubic polynomial? (A symbolic algebra package may be useful here, as well as for the adventure of extending such computations to $\mathrm{Sym}^5 V$, $\mathrm{Sym}^6 V$, and beyond.)