Math 155: Designs and Groups

Homework Assignment #5 (26 March 2010): Synthemes, Totals, and $\Pi_5$

This problem set is due Friday, March 12 in class.

1. a) Fix a 6-element set $\mathbf{6}$. Let $G$ be the graph whose vertices are the 15 synthemes on $\mathbf{6}$ with two synthemes adjacent if and only if they have a pair in common. Show (without invoking $\text{Aut}(S_6)$) that $G$ is strongly regular with the same parameters as the complement of the triangle graph $T_2(\mathbf{6})$ whose vertices are the 2-element subsets of $\mathbf{6}$. Then use an outer automorphism of $S_6$ to show that indeed the two graphs are isomorphic.

   b) The maximal co-cliques of $G$ are the totals (check that again we have equality in the $-sn/(k-s)$ bound). What is the graph obtained from $G$ by deleting such a co-clique and all its incident edges?

2. The remaining exercises outline a proof of the uniqueness of the projective plane of order 5 and the identification of its automorphism group with $\text{PGL}_3(\mathbf{F}_5)$; fill in the details. Recall first that an oval $O$ in such a plane $\Pi_5$ (necessarily of Type I since 5 is odd) is a set of 6 points with no three collinear. Begin by showing that the number of ovals in $\Pi_5$ is $31 \cdot 30 \cdot 25 \cdot 16 \cdot 6 \cdot 1/6! = 3100$. Fix one oval $O$. Of the $5^2 + 5 + 1 = 31$ lines of $\Pi_5$, six are tangent, $\binom{6}{2} = 15$ are secant, and the remaining 10 passant to $O$.

3. Of the $31 - 6 = 25$ points of $\Pi_5 - O$ let $n_i$ be the number lying on $i$ tangents (necessarily $2| i$). As in Prop. 1.48 compute that $\sum_{i>0} \binom{i}{2} n_i = 6 \cdot 5 = 30$ and $\sum_{i>0} \binom{i}{2} n_i = \binom{6}{2} = 15$ to conclude that $n_2 = 15$ and $n_i = 0$ for all $i > 2$. Thus each pair of tangents meets in a different point; we call such a points a “tangential point” or T point for short. Each of the remaining $n_0 = 25 - 15 = 10$ points of $\Pi_5$ lies on no tangent, so is the intersection of three secants to $O$, determining a syntheme. We call such a point a “synthematic point” or S point, and let $S$ be the set of ten synthemes determined by the S points.

4. Any secant meets the tangents to its endpoints on $O$, and the remaining $6-2 = 4$ tangents off $O$. Thus it meets them in pairs at $4/2$ or two T points. Thus the remaining $6 - 2 - 2 = 2$ points on the secant are S points. It follows that each pair occurs twice in the $S$ synthemes, whence $S$ is the complement of a total.
5. Of the 6 lines through each T point, 2 are tangent and 2 secant to \(O\), so the remaining 2 are passant. Thus the number of pairs \(P, l\) with \(P\) a T point and \(l\) a passant line through \(P\) is \(15 \cdot 2 = 30\). But no passant line may go through more than three T points (3 = \(6/2\)): any more than that include two that lie on a common tangent. Thus each of the 10 passant lines contains exactly three T points (and thus three S points since \(6 - 3 = 3\)).

6. Since T points correspond to pairs in \(O\), each passant line determines via its T points three disjoint pairs, i.e. a syntheme. Let \(S'\) be the set of ten synthemes thus arising. We’ll show \(S' = S\). Indeed if \((AB)(CD)(EF)\) is a syntheme not in \(S\) than the T point at the intersection of the \(A, B\) tangents lies on the \(CD, EF\) secants; likewise for \(C, D\). Thus the line joining these two T points is the \(EF\) secant, which does not contain the T point \(E, F\), whence \((AB)(CD)(EF) \notin S'\). Thus \(S' \subseteq S\). But \(\#S' = \#S = 10\) so \(S' = S\) as claimed. [In effect we have also shown that \(\Pi_5\) satisfies the Pappus theorem: secants \(AB, CD, EF\) to an oval \(O\) meet at a point if and only if the corresponding three T points are collinear.]

7. To complete the determination of the structure of \(\Pi_5\) we need only figure out which S points lie on which passant lines. We claim that an S point is on a passant line \(l\) if and only if the corresponding synthemes share exactly one pair (i.e. are adjacent on the graph of Problem 1b). Indeed the three T points on \(l\) are the intersections of \(l\) with \(3 \cdot 2 = 6\) secants; the three S points partition the remaining \(15 - 2 = 9\) secants into 3 synthemes. Show that this can be done uniquely, with the synthemes claimed above. (Probably the simplest way is to actually draw the 9 secants.)

8. We conclude that \(\Pi_5\), if it exists, is determined by a choice of total \(\overline{S}\) on a 6-element set \(O\). Since the six totals on a 6-element set are equivalent, it follows that \(\Pi_5\) is unique up to isomorphism if it exists — which it does because we already know a \(\Pi_5\), namely the algebraic plane \(P^2(F_5)\). Furthermore the number of automorphisms is the number 3100 of ovals times the number \(6! / 6 = 120\) of permutations of \(O\) preserving a total. Check that this is the same as the number of known \(\text{PGL}_3(F_5)\) automorphisms of \(\Pi_5\). Thus \(\text{PGL}_3(F_5)\) is the full automorphism group of \(\Pi_5\). QED