Freshman Seminar 24i: Mathematical Problem Solving
Some induction problems

1. It can be shown[1] that every planar n-gon ($n > 3$) P has an "interior diagonal" — that is, two nonconsecutive vertices V, V' such that the line segment joining V, V' is contained in the interior of P. Use this to prove that the interior angles of P total $(n-2)180°$ (a.k.a. $(n-2)\pi$ radians). [Which version of induction is natural to use here?]

2. Recall that $\binom{n}{k}$ is the binomial coefficient (a.k.a. combinatorial coefficient) defined by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,;$$

if $k < 0$ or $k > n$ we set $\binom{n}{k} = 0$.

i) Given $k \geq 0$ and $n \geq k$, what is $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$?
ii) [somewhat trickier] Given $k \geq 2$ and $n \geq k$, what is

$$\frac{1}{\binom{k}{k}} + \frac{1}{\binom{k+1}{k}} + \frac{1}{\binom{k+2}{k}} + \cdots + \frac{1}{\binom{n}{k}}?$$

3. For $n > 0$ and $d \geq 0$, how many monomials of total degree d are there in n variables? For example, when $d = n = 3$ the number is 10: using variables x, y, z we find the cubic monomials

$$x^3,\ x^2y,\ x^2z,\ xy^2,\ xyz,\ xz^2,\ y^3,\ y^2z,\ yz^2,\ z^3.$$

(In particular the answer is not simply $\binom{n}{d}$: that's the number of degree-d monomials in which no variable appears to power greater than 1.)

4. Give a formula for $(\cos x)\,(\cos(2x))\,(\cos(4x))\,(\cos(8x))\cdots(\cos(2^n x))$. [Especially if you're seeing this for the first time, you might try to use the same trick to evaluate other products such as $\prod_{n=1}^{89}\cos(n°) = \cos(1°)\cos(2°)\cos(3°)\cdots\cos(89°)$, or to construct a problem along the same lines involving $f(x)f(3x)f(9x)f(27x)\cdots f(3^n x)$ for some function f.]

5. [An IMO problem, but it's from the Easiest IMO Ever, and we weren't told there that this was an induction problem ...] Find the integer solution (x, y) of $(x^2 + xy - y^2)^2 = 1$ that has the largest value of $x^2 + y^2$ subject to the conditions $0 \leq x \leq 1981,\ \ 0 \leq y \leq 1981$.

(Follow-up: what can you say about the Diophantine equation[2] $(x^2 + 4xy - y^2)^2 = 1$?)

6. [Thanks to Sonal Jain for suggesting this one] For a S set of n (distinct) positive numbers, let $\Sigma(S) = \{\sum_{t\in T} t \mid T \subseteq S\}$; that is, $\Sigma(S)$ is the set of positive numbers that can be written as the sum of some (possibly empty) subset $T \subseteq S$. Given n, how small can the cardinality $\#(\Sigma(S))$ be? For example, if $n = 1$ or $n = 2$ then all 2^n sums are distinct, and for $n = 3$ there can be at most one coincidence among the 2^3 sub-sums (if the largest element of S is the sum of the other two); so the minimal cardinality is $2, 4, 7$ for $n = 1, 2, 3$ respectively.

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[1] Let v be the left-most vertex (or one of them if there's a choice), and v', v'' its neighbors along the boundary of P. If $v'v''$ is an interior diagonal, we are done. Else there is an interior diagonal vw for some other vertex w in the triangle formed by v, v', v''; for instance, we may choose for w the vertex in that triangle, other than v, v', v'', that is closest to v. Thanks to Zach Abel '10, our resident computational geometer, for finding this construction. Where did we use $n > 3$?

With some more care we can even use this construction to prove that P has an "interior", that is, the fact (which I relegated in class to an application of the Jordan curve theorem) that P splits the plane into exactly two connected regions, an "interior" and an "exterior" of P.

[2] That is, an equation to be solved in integers; Diophantus originally worked with rational numbers, but that can always be encoded into integer solutions as well by replacing an equation in rational numbers $r_1, r_2, \ldots, r_n$ by a homogeneous equation in integers $x_0, x_1, x_2, \ldots, x_n$ where $r_i = x_i/x_0$ for each $i = 1, 2, \ldots, n$.