Freshman Seminar 24i: Mathematical Problem Solving

Some induction problems

1. It can be shown that every planar \( n \)-gon \( (n > 3) \) \( P \) has an “interior diagonal” — that is, two nonconsecutive vertices \( V, V' \) such that the line segment joining \( V, V' \) is contained in the interior of \( P \). Use this to prove that the interior angles of \( P \) total \( (n - 2)180^\circ \) (a.k.a. \( (n - 2)\pi \) radians). [Which version of induction is natural to use here?]

2. Recall that \( \binom{n}{k} \) is the binomial coefficient (a.k.a. combinatorial coefficient) defined by

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} ;
\]

if \( k < 0 \) or \( k > n \) we set \( \binom{n}{k} = 0 \).

i) Given \( k \geq 0 \) and \( n \geq k \), what is \( \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} \)?

ii) [somewhat trickier] Given \( k \geq 2 \) and \( n \geq k \), what is

\[
\binom{1}{k} + \binom{1}{k+1} + \binom{1}{k+2} + \cdots + \binom{1}{n}.
\]

3. For \( n > 0 \) and \( d \geq 0 \), how many monomials of total degree \( d \) are there in \( n \) variables? For example, when \( d = n = 3 \) the number is 10: using variables \( x, y, z \) we find the cubic monomials

\[
x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3.
\]

(In particular the answer is not simply \( \binom{n}{d} \): that’s the number of degree-\( d \) monomials in which no variable appears to power greater than 1.)

4. Give a formula for \( (\cos x)(\cos(2x))(\cos(4x))(\cos(8x))\cdots(\cos(2^n x)) \). [Especially if you’re seeing this for the first time, you might try to use the same trick to evaluate other products such as \( \prod_{n=1}^{89} \cos(n^\circ) = \cos(1^\circ)\cos(2^\circ)\cos(3^\circ)\cdots\cos(89^\circ) \), or to construct a problem along the same lines involving \( f(x)f(3x)f(9x)f(27x)\cdots f(3^n x) \) for some function \( f \).]

5. [An IMO problem, but it’s from the Easiest IMO EVER, and we weren’t told there that this was an induction problem ...] Find the integer solution \( (x, y) \) of \( (x^2 + xy - y^2)^2 = 1 \) that has the largest value of \( x^2 + y^2 \) subject to the conditions \( 0 \leq x \leq 1981, \ 0 \leq y \leq 1981 \).

(Follow-up: what can you say about the Diophantine equation \( (x^2 + 4xy - y^2)^2 = 1? \))

6. [Thanks to Sonal Jain for suggesting this one] For a \( S \) set of \( n \) (distinct) positive numbers, let \( \Sigma(S) = \{ \sum_{t \in T} t \mid T \subseteq S \} \); that is, \( \Sigma(S) \) is the set of positive numbers that can be written as the sum of some (possibly empty) subset \( T \subseteq S \). Given \( n \), how small can the cardinality \( \#(\Sigma(S)) \) be? For example, if \( n = 1 \) or \( n = 2 \) then all \( 2^n \) sums are distinct, and for \( n = 3 \) there can be at most one coincidence among the \( 2^3 \) sub-sums (if the largest element of \( S \) is the sum of the other two); so the minimal cardinality is 2, 4, 7 for \( n = 1, 2, 3 \) respectively.

\[1\] Let \( v \) be the left-most vertex (or one of them if there’s a choice), and \( v', v'' \) its neighbors along the boundary of \( P \). If \( v'v'' \) is an interior diagonal, we are done. Else there is an interior diagonal \( vw \) for some other vertex \( w \) in the triangle formed by \( v, v', v'' \); for instance, we may choose for \( w \) the vertex in that triangle, other than \( v, v', v'' \), that is closest to \( v \). Thanks to Zach, our resident computational geometer, for finding this construction. Where did we use \( n > 3? \)

With some more care we can even use this construction to prove that \( P \) has an “interior”, that is, the fact (which I relegated in class to an application of the Jordan curve theorem) that \( P \) splits the plane into exactly two connected regions, an “interior” and an “exterior” of \( P \).

\[2\] That is, an equation to be solved in integers; Diophantus originally worked with rational numbers, but that can always be encoded into integer solutions as well by replacing an equation in rational numbers \( r_1, r_2, \ldots, r_n \) by a homogeneous equation in integers \( x_0, x_1, x_2, \ldots, x_n \) where \( r_i = x_i/x_0 \) for each \( i = 1, 2, \ldots, n. \)