Some simple applications:

1) Of all rectangular boxes of volume $V$, the cube has the smallest surface area.

2) An “open box” is a rectangular box missing one of its six sides (so that we can put things in and out of the box). Given $V$, what is the open box of volume $V$ that has minimal surface area? Upon further reflection, can you derive this directly from (1)?

3) Of all triangles of area $A$, the equilateral triangle has the smallest perimeter.

Some proofs of the inequality and generalizations:

4) Say a real-valued function $f$ on some (possibly infinite) interval $I$ is “midpoint-convex” upwards if

$$f\left(\frac{1}{2}(x + y)\right) \leq \frac{1}{2}\left(f(x) + f(y)\right)$$

for all $x, y$ in $I$; we say $f$ is strictly midpoint-convex upwards if equality holds only for $x = y$. Prove that if $f$ is midpoint-convex upwards then Jensen’s inequality

$$f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \leq \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)$$

holds for all $n$ and for all $x_1, \ldots, x_n$ in $I$; and that if $f$ is strictly midpoint-convex then equality holds if and only if all the $x_i$ are equal. [Hint: the inequality is trivial for $n = 1$ and the definition of midpoint-convexity for $n = 2$. Prove it first for $n = 4, 8, 16, \ldots$; to get the general case, try first to derive the $n = 3$ case from $n = 4$. Thanks to Zach for reminding me of the AM–GM case of this trick.]

5) Show directly (without calculus etc.) that the functions $x^2$ and $e^x$ are midpoint-convex upwards on all of $\mathbb{R}$, that $1/x$ is midpoint-convex upwards on $x > 0$, and that $\sin x$ is midpoint-convex downwards on $0 \leq x \leq \pi$. (Hint: it can be surprisingly useful that $a \geq b$ if and only if $a - b \geq 0$.) Jensen’s inequality for these functions then follows from the previous problem.

6) Suppose $w_1, w_2, \ldots, w_n$ are nonnegative numbers with $w_1 + w_2 + \cdots + w_n = 1$. The weighted average of $z_1, z_2, \ldots, z_n$ with weights $w_1, w_2, \ldots, w_n$ is $w_1z_1 + w_2z_2 + \cdots + w_nz_n$. (For instance, the usual weighted average is recovered by setting each $w_i$ equal $1/n$, and the barycentric coordinates of a point $P$ in triangle $ABC$ are the weights for which $P$ is the weighted average of $A, B, C$ — that’s one reason I didn’t restrict $z_1, \ldots, z_n$ to real numbers.) Jensen’s inequality for weighted averages states that

$$f(w_1x_1 + w_2x_2 + \cdots + w_nx_n) \leq w_1f(x_1) + w_2f(x_2) + \cdots + w_nf(x_n)$$

if $f$ is convex upwards, and likewise

$$f(w_1x_1 + w_2x_2 + \cdots + w_nx_n) \geq w_1f(x_1) + w_2f(x_2) + \cdots + w_nf(x_n)$$

if $f$ is convex downwards. (Again the inequality compares the function of the average with the average of the function.) Prove this inequality: (i) in the same graphical way that we did for the unweighted version; (ii) as a consequence of Jensen (first do the case of rational $w_i$, then use continuity). If $f$ is strictly convex, when does equality hold?

7) Show that weighted Jensen still reduces to Cauchy–Schwarz for $f(x) = x^2$ or $f(x) = 1/x$.

8) Suppose $P$ is a polynomial with positive coefficients. Prove that $P(x)P(y) \geq (P(\sqrt{xy}))^2$ for all $x, y > 0$. When does equality hold? What are the functions whose convexity you can deduce from that inequality?

9) Suppose $f(x)$ is convex downwards on $a \leq x \leq b$ and we fix some $n \geq 1$ and $s$ between $na$ and $nb$. Consider the sum $f(x_1) + \cdots + f(x_n)$ subject to $a \leq x_i \leq b$ and $x_1 + \cdots + x_n = s$. Jensen tells us that this is maximized when each $x_i$ equals $s/n$. What choice of $x_1, \ldots, x_n$ minimizes the sum? For instance, how small can $\sin \alpha + \sin \beta + \sin \gamma + \sin \delta + \sin \epsilon$ get if $0 \leq \alpha, \beta, \gamma, \delta, \epsilon \leq \pi/2$ and $\alpha + \beta + \gamma + \delta + \epsilon = 5$?