A p-adic Jacquet-Langlands Correspondence

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Abstract

In this paper, we construct a candidate p-adic Jacquet-Langlands correspondence. This is a correspondence between unitary continuous admissible representaitons of $\mathrm{GL}_2(\mathbb{Q}_p)$ valued in p-adic Banach spaces, and unitary continuous representations of $D^\times$ valued in p-adic Banach spaces. Here, D is the quaternion algebra over $\mathbb{Q}_p$. The characterizing properties that are shown are a local-global compatability and a calculation of the locally algebraic vectors.

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1. Introduction and Main Results

Let $D/\mathbb{Q}_p$ be the unique quaternion algebra, and let $E/\mathbb{Q}_p$ be a finite extension that is "large enough". Classical work of Jacquet and Langlands allows one to classify smooth representations of $D^\times$ in terms of smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Recently, there has been a lot of interest in a different class of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, namely continuous unitary representations valued in Banach spaces over E. The goal in this paper is to construct an analogue of the Jacquet-Langlands correspondence for continuous unitary representations valued in E-Banach spaces. The following theorem shows that there is an analogue of the classical theory, a "p-adic Jacquet-Langlands correspondence."

Theorem 1.0.1. *Let π be a continuous unitary irreducible admissible representation of $GL_2(\mathbb{Q}_p)$ in an E-Banach space. Then there is a continuous unitary representation $J(\pi)$ of $D^\times$ valued in an E-Banach space.*

The representation $J(\pi)$ is constructed purely locally. The above theorem is vacuous without any properties of $J(\pi)$, and the following theorems are meant to give some characterizing properties of $J(\pi)$.

Let $\Delta/\mathbb{Q}$ be a division algebra such that $\Delta_{\mathbb{Q}_p} = D$ and $\Delta_{\mathbb{R}} = M_2(\mathbb{R})$. Let $G = \Delta^\times$ as an algebraic group over $\mathbb{Q}$ and S_G be the set of all primes $\ell \neq p$ such that $G(\mathbb{Q}_\ell) \neq \mathrm{GL}_2(\mathbb{Q}_\ell)$. We will assume from here on out that $\ell \not\equiv \pm 1 \pmod{p}$ for all $\ell \in S_G$. Now, let S_0 be a finite set of primes disjoint from S_G, and let $S = S_0 \cup \{p\}$. Finally, let $G_{S_0} = \prod_{\ell \in S_0} G(\mathbb{Q}_\ell)$. Choose a maximal compact $K_0^S \subset G(\mathbb{A}_f^S)$, and to any compact open subgroup $K_p \times K_{S_0} \subset D^\times \times G_{S_0}$, we can associate a Shimura curve $Sh_{K_p K_{S_0} K_0^S}/\mathbb{Q}$. Following Emerton, define the completed cohomology as follows:

Definition 1.0.2 (Completed Cohomology)**.** *The completed cohomology for G, denoted $\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0})$ is defined as*

$$\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0}) = \left(\varprojlim_s \varinjlim_{K_p} H^1_{\acute{e}t}(Sh_{K_pK_{S_0}K_0^S,\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s) \right).$$

Also, let $\hat{H}^1_{E,G}(K_{S_0}) = \hat{H}^1_{\mathcal{O}_E,G}(K_{S_0}) \otimes_{\mathcal{O}_E} E$. This is a Banach space over E with unit ball given by $\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0})$. Finally, let

$$\hat{H}^1_{*,G,S} = \varinjlim \hat{H}^1_{*,G}(K_{S_0})$$

where $*$ is either $\mathcal{O}_E$ or E, and the limit is taken over all compact open subgroups $K_{S_0} \subset G_{S_0}$.

There are commuting unitary actions of $G_{\mathbb{Q}}$, $D^\times$, and a Hecke algebra $\mathbb{T}$ on $\hat{H}^1_{*,G}(K_{S_0})$ and $\hat{H}^1_{*,G,S}$. Additionally, there is an action of G_{S_0} on $\hat{H}^1_{*,G,S}$ that commutes with all of the aforementioned actions. Let ρ be a 2-dimensional continuous representation of $G_{\mathbb{Q}}$ that is unramified at all places $\ell \notin S \cup S_G$. Assume further that ρ is promodular (conjecturally odd), that $\overline{\rho}|_{G_{\mathbb{Q}_\ell}} = \left(\begin{smallmatrix} \chi & * \\ & \chi\epsilon \end{smallmatrix}\right)$ for all $\ell \in S_G$ (ϵ is the cyclotomic character, and $*$ is non-zero in this case), and that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is neither $\left(\begin{smallmatrix} \chi & * \\ & \chi\overline{\epsilon} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \chi & * \\ & \chi \end{smallmatrix}\right)$, nor $\left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ (where χ is any mod p character of $G_{\mathbb{Q}_p}$, $\overline{\epsilon}$ is the mod p cyclotomic character, and $*$ may be zero or nonzero in this case). Let $\pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})$ be the representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ associated to $\rho|_{G_{\mathbb{Q}_\ell}}$ under the local Langlands correspondence (see appendix A for a precise definition of π_{LL}).

Theorem 1.0.3. *With the above assumptions on ρ, there is a representation $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ such that*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G,S}) \cong J(B(\rho|_{G_{\mathbb{Q}_p}})) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho|_{\mathbb{Q}_\ell}).$$

Here, $B(\rho|_{G_{\mathbb{Q}_p}})$ is the p-adic Langlands correspondence that is discussed in appendix [**?**]. The key point of this theorem is that the representation $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ depends only on the local Galois representation $\rho|_{G_{\mathbb{Q}_p}}$. The fact that it doesn't depend on all of ρ, or even any of the choices made in constructing G is the main thrust of the theorem. Additionally, in the above isomorphism, there is an action of $\mathbb{T}$ on $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho, \hat{H}^1_{E,G,S})$. Importantly in the above theorem, one has that T_ℓ acts via $\mathrm{tr}(\mathrm{Frob}_\ell|_\rho)$; this is a compatability for the Hecke algebra.

A natural question to ask is "What are the locally algebraic vectors in $J(B(\rho|_{G_{\mathbb{Q}_p}}))$?" More precisely, assume that $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$. Then one can construct a Weil-Deligne represntation $WD_{\rho|_{G_{\mathbb{Q}_p}}}$. If $WD_{\rho|_{G_{\mathbb{Q}_p}}}$ is indecomposable, then there is an associated smooth admissible $D^\times$ representation Sm given by using the local Langlands and Jacquet-Langlands correspondences on $WD_{\rho|_{G_{\mathbb{Q}_p}}}$. Additionally, the weights $w_1 + 1$ and w_2 are weights for an algebraic representation Alg of $D^\times$. This construction is similar to the one performed in [4].

Theorem 1.0.4. *Let ρ be as in Theorem 1.0.3. If the assumptions on $\rho|_{G_{\mathbb{Q}_p}}$ in the above construction hold, then the space of locally algebraic vectors $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \cong Sm \otimes Alg$. If one of the assumptions does not hold, then $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \cong 0$.*

Some more interesting notes about the locally algebraic vectors: following through the construction of the locally algebraic vectors, one sees that there are no locally algebraic vectors for crystalline representations. Additionally, the locally algebraic vectors are still finite dimensional for a fixed representation $J(B(\rho|_{G_{\mathbb{Q}_p}}))$, and so they are closed in $J(B(\rho|_{G_{\mathbb{Q}_p}}))$. However, $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ is believed to be infinite dimensional for all ρ. Thus, even when the associated Galois representation $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible, one should have a proper closed subrepresentation of $J(B(\rho|_{G_{\mathbb{Q}_p}}))$. Thus, one would not expect a statement like $J(\pi)$ is "as irreducible as" π to be true. This also suggests that $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ is not directly characterized by the space of locally algebraic vectors together with a continuous unitary admissible norm on said space.

There is an auxillary group $\overline{G}/\mathbb{Q}$ that arises in the proof of these theorems. This group is an inner form of G with invariants at p and ∞ swapped. Since $\overline{G}(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$, one can identify subgroups of $\overline{G}(\mathbb{A}_f^p)$ with subgroups of $G(\mathbb{A}_f^p)$. Continuous functions on the double coset space $X_{\overline{K}_{S_0}} := \overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}_f)/\overline{K}_{S_0}\overline{K}_0^S$ valued in E gives an analogue of the completed cohomology of G for $\overline{G}$.

The idea behind the proof of Theorem 1.0.3 is as follows: there is a p-adic analytic uniformization (called the Cerednik-Drinfel'd uniformization) $\Sigma^n \times X_{\overline{K}_{S_0}}/\mathrm{GL}_2(\mathbb{Q}_p) \cong Sh_{K_p^n K_{S_0} K_0^S, \mathbb{C}_p}$, where Σ^n is a cover of the Drinfel'd upper half plane. The plan is thus to first analyze the space of continuous functions $\mathcal{C}^0(X_{\overline{K}_{S_0}}, E)$, and then apply that knowledge to understanding what the uniformization says about completed cohomology. It turns out that the Hecke algebra $\mathbb{T}$ will act on the space of automorphic forms for $\overline{G}$, and so the first step is to describe this space as a $\mathbb{T}[\mathrm{GL}_2(\mathbb{Q}_p)]$-module. This can be done by an adaptation of the main argument in [15]. There is a spectral sequence relating the cohomology of $Sh_{K_p^n K_{S_0} K_0^S}$ to that of Σ^n and $X_{\overline{K}_{S_0}}$, and analysis of this spectral sequence will give Theorem 1.0.3.

As for Theorem 1.0.4, the argument is an application of the results of [14]. That paper gives a spectral sequence relating the locally algebraic vectors to the cohomology of various local systems $\mathcal{V}_W$ on $Sh_{K_p^n K_{S_0} K_0^S}$. Since the cohomology of these local systems is well understood, the only thing that is needed to compute the locally algebraic vectors is to perform an analysis of the spectral sequence.

There is a more general version of the above theorems. Let $F/\mathbb{Q}$ be a totally real field with one place v/p. If π is a representation of $\mathrm{GL}_2(F_v)$, then one can construct a representation $J'(\pi)$ of $D_{F_v}^\times \times G_{F_v}$. Again, this construction is purely local and depends only on F_v and not on any other choices. Now, one considers a unitary group G/F that is D_{F_v} at F_v, $U(1,1)$ at exactly one infinite place, and $U(2)$ at every other infinite place. There is a Shimura curve $Sh_{K_v K^v}/F$, and one may talk about $\hat{H}^1_{E,G}(K^v)$, as before. Additionally, there is an auxiliary group $\overline{G}$ with $\overline{G}(F) = \mathrm{GL}_2(F)$ that arises in the uniformization of $Sh_{K_v K^v}$.

In this situation, there is the following weaker version of Theorem 1.0.3:

Theorem 1.0.5. *If π arises as a representation of $\overline{G}$, then $J'(\pi)$ arises in $\hat{H}^1_{E,G}(K^v)$.*

The proof of Theorem 1.0.5 follows the same lines as the proof of Theorem 1.0.3, with the weakening coming from the fact that there is no p-adic Langlands correspondence to understand the space $\mathcal{C}^0(X_{\overline{K_{S_0}}}, E)$ as there was in the $F_v = \mathbb{Q}_p$ case.

2. The Drinfel'd Upper Half Plane

This section will introduce the local object that will be important throughout the rest of the paper. As a rigid space, the Drinfel'd upper half plane $\Omega^2_{F_v}/\widehat{F^{ur}_v}$ (which we will use Ω for for the rest of this section) is just $\mathbb{P}^1 \backslash \mathbb{P}^1(F_v)$. This is naturally a rigid analytic variety. For the rest of this article, Ω will be viewed as an adic space, primarialy due to the theory of étale cohomology for adic spaces.

This exposition will be based on the exposition in [6]. The results that are discussed were originally proved in [11], [13], and [12], and there is a vast generalization that can be found in [20].

2.1. **The Moduli Interpretation and the Level Covers.** A result of Deligne (which can be found in [12]) shows that $\Omega^2_{F_v}$ has the structure of a formal scheme. To define the moduli problem that $\breve{\Omega}$ represents, recall that D_{F_v} is the unique quaternion algebra over F_v, and let $\nu : D_{F_v} \to F_v$ be the reduced norm. Let $\mathcal{M}$ be the functor associating to an $\mathcal{O}_{\widehat{F^{ur}_v}}$-scheme S with $\varpi_{\widehat{F^{ur}_v}}\mathcal{O}_S$ locally nilpotent (such a scheme S is said to be in the category $\mathrm{Nilp}_{\mathcal{O}_{\widehat{F^{ur}_v}}}$) the set of triples $(G, \iota_{D_{F_v}}, \varrho)$. Here G is a two-dimensional formal $\mathcal{O}_{F_v}$-module over S with F_v-height four, $\iota_{D_{F_v}} : \mathcal{O}_{D_{F_v}} \hookrightarrow \mathrm{End}(X)$ gives an action of $\mathcal{O}_{D_{F_v}}$ on G. $\iota_{D_{F_v}}$ is assumed to satisfy the following condition: let F'_v be the unramified quadratic extension of F_v. Then $\mathcal{O}_{F'_v} \hookrightarrow \mathcal{O}_{D_{F_v}}$, so, via $\iota_{D_{F_v}}$, one gets an action of $\mathcal{O}_S \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{F'_v}$ on $\mathrm{Lie}(G)$. The assumption is that this makes $\mathrm{Lie}(G)$ a locally free sheaf of rank one over $\mathcal{O}_S \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{F'_v}$. Finally, over $\overline{k}_{F_v}$, there is only one two-dimensional formal $\mathcal{O}_{F_v}$-module $\mathbb{G}$ of height 4 with an action of $\mathcal{O}_{D_v}$. Then $\varrho : G \times_S \overline{S} \to \mathbb{G} \times_{\overline{k_{F_v}}} \overline{S}$ is chosen to be a quasi-isogeny of height zero.

Theorem 2.1.1 (Drinfel'd)**.** *The functor $\mathcal{M}$ is represented by the formal scheme $\breve{\Omega}$.*

There is a universal formal $\mathcal{O}_{F_v}$-module $\mathscr{G} \to \Omega$. Inside $\mathscr{G}$ is the universal $\varpi^n_{D_{F_v}}$-torsion, written $\mathscr{G}[\varpi^n_{D_{F_v}}]$. $\mathscr{G}[\varpi^n_{D_{F_v}}]$ is a free $\mathcal{O}_{D_{F_v}}/\varpi^n_{D_{F_v}}$-module over Ω of rank one. There are maps $\mathscr{G}[\varpi^n_{D_{F_v}}] \hookrightarrow \mathscr{G}[\varpi^{n+1}_{D_{F_v}}]$ and $\mathscr{G}[\varpi^{n+1}_{D_{F_v}}] \twoheadrightarrow \mathscr{G}[\varpi^n_{D_{F_v}}]$. The first is the natural inclusion, and the second is multiplication by $\varpi_{D_{F_v}}$. Let $\Sigma^n = \mathscr{G}[\varpi^n_{D_{F_v}}] \backslash \mathscr{G}[\varpi^{n-1}_{D_{F_v}}]$ for $n > 0$. As a note, Σ^n is the first object in this paragraph that is not naturally a formal scheme, as the removal of $\mathscr{G}[\varpi^{n-1}_{D_{F_v}}]$ cannot be done integrally. Multiplication by $\varpi_{D_{F_v}}$ gives rise to maps $\Sigma^{n+1} \to \Sigma^n$. We will let $\Sigma = \varprojlim \Sigma^n$. The only operation that will be done to Σ is taking cohomology, by which it is meant that $H^i(\Sigma, *) = \varinjlim H^i(\Sigma^n, *)$.

2.2. **Group Actions.** There are three different groups that act on Σ. The first group that will be discussed is $\mathrm{GL}_2(F_v)$. PGL_2 is the group of automorphisms of $\mathbb{P}^1$, and so there is a natural action of GL_2 on $\mathbb{P}^1$. Since $\mathbb{P}^1(F_v)$ is a closed F_v-orbit of the action of $\mathrm{GL}_2(F_v)$, the action of $\mathrm{GL}_2(F_v)$ on $\mathbb{P}^1$ preserves Ω. This action, however, doesn't have a moduli theoretic interpretation, and so will not naturally extend to $\mathscr{G}$ (and thus to Σ^n). If one twists the action by $g \to \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g))}$, where $\mathrm{Frob}_{F_v} : \widehat{F_v^{ur}} \to \widehat{F_v^{ur}}$ is geometric Frobenius, one gets another action of $\mathrm{GL}_2(F_v)$ on Ω. This action doesn't preserve the structure morphism $\Omega \to \widehat{F_v^{ur}}$. This is well-defined, as the original action did preserve the structure morphism, and so one has that, letting $\cdot$ be used for the original $\mathrm{GL}_2(F_v)$-action on Ω, $\mathrm{Frob}_{F_v}(g \cdot x) = g \cdot \mathrm{Frob}_{F_v}(x)$ and thus $g_1 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_1))}(g_2 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_2))}(x)) = g_1 \cdot g_2 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_1 g_2))}(x)$. This new action has a moduli interpretation ([11] and [13] have the details), and thus extends to $\mathscr{G}$ (and thus to Σ^n and thus to Σ).

Since $\Sigma^n \to \Omega$ is an $(\mathcal{O}_{D_{F_v}}/\varpi^n_{D_{F_v}})^\times$-torsor, one gets an action of $\mathcal{O}^\times_{D_{F_v}}$ on Σ^n. It is straightforward to see that the maps $\Sigma^n \to \Sigma^m$ are $\mathcal{O}^\times_{D_{F_v}}$-equivariant, and thus one has an action of $\mathcal{O}^\times_{D_{F_v}}$ on Σ. If (x_n) is a sequence of points $x_n \in \Sigma^n$ with $\varpi_{D_{F_v}} x_n = x_{n-1}$ (i.e. (x_n) is a point of Σ), then it is natural to define $\varpi_{D_{F_v}} \cdot (x_n) = (x_n)$. But, as before, the "correct" action is not this one, but rather this one twisted by $d \mapsto \mathrm{Frob}_F^{v_F(\nu(d))}$. This action of $D^\times_{F_v}$ commutes with the aforementioned action of $\mathrm{GL}_2(F_v)$, giving rise to an action of $\mathrm{GL}_2(F_v) \times D^\times$ on Σ.

While the action of $\mathrm{GL}_2(F) \times D^\times$ doesn't respect the structure morphism $\Sigma \to \widehat{F_v^{ur}}$, it does respect the the morphism $\Sigma \to \widehat{F_v^{ur}} \to F_v$. Thus, one gets an action of G_{F_v} on $H^i(\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}(\Sigma) \times_{F_v} \mathbb{C}_p, *)$ that automatically commutes with the action of $\mathrm{GL}_2(F_v) \times D^\times_{F_v}$ (Here, Res is standard restriction of scalars). Going a little deeper into this action, one has that $\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}(\Sigma) \times_{F_v} \mathbb{C}_p$ is the union of $\hat{\mathbb{Z}}$ copies of $\Sigma \times_{\widehat{F_v^{ur}}} \mathbb{C}_p$, with an element $g \in G_{F_v}$ shifting the index based on the image of g in $G_{k_{F_v}} = \mathrm{Frob}_{F_v}^{\hat{\mathbb{Z}}}$.

For $(g, d) \in \mathrm{GL}_2(F_v) \times D^\times$, choose a $w \in G_{F_v}$ such that $w|_{G_{k_{F_v}}} = \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g)\nu(d))}$. From the definition of the above action, one has $(g, d) : \Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p \to \left(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p\right)^w$ is an isomorphism (Here, if X is an adic space over $\mathbb{C}_p$, X^w is X twisted by w). In summary, one has the following:

Theorem 2.2.1. *The group* $\mathrm{GL}_2(F_v) \times D^\times_{F_v} \times G_{F_v}$ *acts on* $\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}(\Sigma) \times_{F_v} \mathbb{C}_p$.

2.3. **Connected Components.** This version of the space Σ^n is not geometrically connected. In order to describe the connected components, we need to introduce a group $P \subset \mathrm{GL}_2(F_v) \times D^\times_{F_v} \times G_{F_v}$ defined by $P = \{(m, d, g) \in \mathrm{GL}_2(F_v) \times D^\times_{F_v} \times G_{F_v} | \det(m)\nu(d)\mathrm{cl}(g)^{-1} \in \mathcal{O}^\times_{F_v}\}$. Here, cl is the Artin map of local class field theory, normalized so that geometric Frobenius goes to an element of valuation 1. The goal will be to describe the connected components not just as a set, but also with an action of P. Moreover, P is the largest subgroup of $\mathrm{GL}_2(F_v) \times D^\times_{F_v} \times G_{F_v}$ that could reasonably be expected to act on the connected components of Σ^n, as this is the subgroup that preserves the morphism $\Sigma^n \to \widehat{F_v^{ur}}$.

Proposition 2.3.1. *There is an identification*

$$\varprojlim_n \pi_0(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p) \cong \mathcal{O}_{F_v}^\times.$$

For $(m, d, g) \in P$*, the action on* $\varprojlim \pi_0(\Sigma^n)$ *is given by multiplication by the number* $\det(m)\nu(d)\mathrm{cl}(g)^{-1}$ *(which is guarenteed to be a unit due to the definition of* P*).*

The proof of this result will be outlined at the end of the next section.

3. The Global Picture

Choose a number field $F/\mathbb{Q}$ such that F is totally real, with real places $v_{\infty,1}, \dots, v_{\infty,d}$, and there is exactly one plave v over p. If $F \neq \mathbb{Q}$, we need to propertly define the unitary group mentioned in the introduction. Choose a CM extension L/F such that v splits as $v_1 v_2$ in L. Then choose a quaternion algebra Δ/L such that $\Delta(L_{v_1}) = \Delta(L_{v_2}) = D_{F_v}$, and such that there is an involution i of the second kind with signature $(1,1)$ at $v_{\infty,1}$ and $(2,0)$ at $v_{\infty,j}$ for $j > 1$. Then $G = \{d \in \Delta | d \cdot i(d) = 1\}$, an algebraic group over F. The assumptions listed imply that $G(F_v) = D_{F_v}^\times$, $G(F_{v_{\infty,1}}) = U(1,1)$, and $G(F_{v_{\infty,j}}) = U(2)$ for $j > 1$. If $K \subset G(\mathbb{A}_{F,f})$ is a compact open subgroup, then there is a unitary Shimura curve Sh_K/L. It will be convienent to define $K_v^n = \{d \in \mathcal{O}_{D_{F_v}}^\times | d \equiv 1 \pmod{\varpi_{D_{F_v}}^n}\}$ and to choose $K = K_v^n K^v$ with K^v a compact open subgroup of $G(\mathbb{A}_{F,f}^v)$. The goal of this section is to give a p-adic analytic uniformization of $Sh_{K_v^n K^v}$.

3.1. **The Group $\overline{G}$.** To that end, we will introduce another group $\overline{G}$ over F. If $F = \mathbb{Q}$, let $\overline{\Delta}/\mathbb{Q}$ be the division algebra that has the same invariants as Δ away from p and ∞, and is now $M_2(\mathbb{Q}_p)$ at p and Hamilton's quaternions H at ∞. Define $\overline{G} = \overline{\Delta}^\times$ as an algebraic group over $\mathbb{Q}$. If $F \neq \mathbb{Q}$, this group is an inner form of G

that is isomorphic to G away from v and $v_{\infty,1}$, is GL_2 at v, and $U(2)$ at $v_{\infty,1}$. Visibly, one may identify $G(\mathbb{A}_{F,f}^v) \cong \overline{G}(\mathbb{A}_{F,f}^v)$ in both cases. Choose an isomorphism that comes from an algebra anti-isomorphism of the underlyling algebras. Then one may identify K^v with a subgroup $\overline{K}^v$ in $\overline{G}(\mathbb{A}_{F,f}^v)$.

The theory of automorphic forms for $\overline{G}$ is related to the double coset spaces $\overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}_f$ where $\overline{K}_f$ is a compact open subgroup of $\overline{G}(\mathbb{A}_{F,f})$. These spaces are finite sets of points which are sometimes called "Hida varieties." In this paper, the spaces $X_{\overline{K}^v} := \overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}^v$ will also be relevant. This space is a compact $\mathrm{GL}_2(F_v)$-set and functions on this space will come from automorphic forms with any level at p. Additionally, $X_{\overline{K}^v}$ breaks up into finitely many orbits under the $\mathrm{GL}_2(F_v)$ action. The orbits are parameterized by the finite set $\overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f}^v)/\overline{K}^v$, and if one chooses a set of double coset representatives $\overline{g}_i$, these orbits are of the form $\Gamma_i\backslash\mathrm{GL}_2(F_v)\overline{g}_i$, where $\Gamma_i = \{\gamma \in \overline{G}(F)|\overline{g}_i\gamma\overline{g}_i^{-1} \in \overline{K}^v\}$ (inclusion is under the natural map from $\overline{G}(F) \to \overline{G}(\mathbb{A}_{F,f}^v)$). The Γ_is are discrete cocompact subgroups of $\mathrm{GL}_2(F_v)$.

3.2. **Čerednik-Drinfel'd Uniformization.** The main result is the following:

Theorem 3.2.1 (Čerednik-Drinfel'd Uniformization)**.** *With notation as above, there is an isomorphism*

$$(Sh_{K^vK_v^n} \times_F \mathbb{C}_p)^{an} \cong \left(\left(\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}\Sigma^n \times_{F_v} \mathbb{C}_p\right) \times X_{\overline{K}^v}\right)/GL_2(F_v).$$

A few remarks are in order. Since there is a decomposition $X_{\overline{K}^v} = \coprod \Gamma_i\backslash\mathrm{GL}_2(F_v)$, the right hand side of the isomorphism may be written as $\coprod \Gamma_i\backslash\left(\mathrm{Res}_{F_v}^{\widehat{F_v{}^{ur}}}\Sigma^n \times_{F_v} \mathbb{C}_p\right)$. Additionally, if $\Gamma_i' = \Gamma_i \cap \{g \in \mathrm{GL}_2(F_v)|\det(g) \in \mathcal{O}_{F_v}^\times\}$ and $n_i = [\Gamma_i/\Gamma_i', v_{F_v}(F_v^\times)]$, then there is an isomorphism $\Gamma_i\backslash\left(\mathrm{Res}_{F_v}^{\widehat{F_v{}^{ur}}}\Sigma^n \times_{F_v} \mathbb{C}_p\right) \cong \coprod_{j=0}^{n_i-1}\left(\Gamma_i'\backslash(\Sigma^n \times_{\widehat{F_v{}^{ur}}} \mathbb{C}_p)\right)^{\mathrm{Frob}_{F_v}^j}$. In order to give a G_{F_v} action on $\coprod_{j=0}^{n_i-1}\left(\Gamma_i'\backslash(\Sigma^n \times_{\widehat{F_v{}^{ur}}} \mathbb{C}_p)\right)^{\mathrm{Frob}_{F_v}^j}$, one can give a model of $\coprod_{j=0}^{n_i-1}(\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^j}$ over F_v. One thus needs to give a Frob_{F_v}-semilinear map $\varphi : \coprod_{j=0}^{n_i-1}(\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^j} \to \coprod_{j=0}^{n_i-1}(\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^j}$. If $j < n_i - 1$, then define

$\varphi : (\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^j} \to (\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^{j+1}}$ to just be Frobenius. If $j = n_0 - 1$, choose $g \in \Gamma_i$ such that $v_{F_v}(\det(g)) = n_i$, and define $\varphi : (\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^{n-1}} \to (\Gamma_i'\backslash\Sigma^n)$ to be $g^{-1}\cdot\mathrm{Frob}_{F_v}$. It is easy to see that this is isomorphic to $\Gamma_i\backslash\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}\Sigma^n$ over F_v.

Additionally, there are Hecke operators acting on both the Shimura curve, and the set $X_{\overline{K}^v}$. The above isomorphism is an isomorphism of analytic varieties, together with an action of G_{F_v} and of the Hecke operators. The action of all of $G_{\mathbb{Q}}$ or G_L in the $F \neq \mathbb{Q}$ case is not completely lost, as one must have that there is compatability between the Hecke operators and Galois representation.

The Čerednik-Drinfel'd uniformization is what is needed to prove proposition 2.3.1, which describes the connected components of Σ^n. The main idea is to use the reciprocity law for Shimura varieties to describe the connected components of $Sh_{K^vK_v^n}$. Then, the Čerednik-Drinfel'd uniforimzation lets you relate the connected components of $Sh_{K^vK_v^n}$ to those of Σ^n and $X_{\overline{K}^v}$, and this gives the result. A full proof may be found in [2].

4. Local-Global Compatibility for $\overline{G}$ over $\mathbb{Q}$

Let S_0 be a finite set of places of F not containing v nor any infinte place, nor any place where $\overline{G}$ is nonsplit. The goal of this section is to understand the space $X_{\overline{K}_{S_0}} := \overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}_{S_0}\overline{K}_0^S$. It is a well-known fact that $X_{\overline{K}_{S_0}}$ is a compact $\mathrm{GL}_2(F_v)$-set, which is equivalent to the pair of facts that $X_{\overline{K}_{S_0}}/\overline{K}_v$ is finite for all compact open subgroups $\overline{K}_v \subset \mathrm{GL}_2(F_v)$ and that $X_{\overline{K}_{S_0}} = \varprojlim_{\overline{K}_v} X_{\overline{K}_{S_0}}/\overline{K}_v$. The ideas in this section are inspired heavily by chapters 5 and 6 of [15]. Additionally, [7] and [8] are invaluable as references, especially to highlight some of the simplifications that arise in this section.

4.1. **The Completed H^0 for $X_{\overline{K}_{S_0}}$.** The main object of study in this section is the Banach space

$$\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}) := \left(\varprojlim_s \varinjlim_{\overline{K}_v} H^0(X_{\overline{K}_{S_0}}/\overline{K}_v, \mathcal{O}_E/\varpi_E^s) \right) \otimes_{\mathcal{O}_E} E,$$

and its cousin $\hat{H}^0_{E,\overline{G},S} = \varinjlim_{\overline{K}_{S_0}} \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$.

Proposition 4.1.1. $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}^0_E(X_{\overline{K}_{S_0}})$*, where* $\mathcal{C}^0_E(X_{\overline{K}_{S_0}})$ *is the space of continuous* E*-valued functions on* $X_{\overline{K}_{S_0}}$ *endowed with the sup-norm.*

Proof. Let $f : X_{\overline{K}_{S_0}} \to E$ be a continuous function. Since $X_{\overline{K}_{S_0}}$ is compact, there is an integer i such that f factors as $X_{\overline{K}_{S_0}} \xrightarrow{f'} \varpi_E^i \mathcal{O}_E \hookrightarrow E$. Then f is continuous if and only if f' is continuous, and the latter is equivalent to the functions $f'_j : X_{\overline{K}_{S_0}} \to \varpi_E^i \mathcal{O}_E \to \varpi_E^i \mathcal{O}_E/\varpi_E^{i+j}\mathcal{O}_E$ being continuous for all j.

The set $\varpi_E^i \mathcal{O}_E/\varpi_E^{i+j}\mathcal{O}_E$ is discrete, so f'_j is continuous if and only if it is locally constant. While local constancy is with respect to the natural topology of $X_{\overline{K}_{S_0}}$, a basis of open neighborhoods of a point x is just given by the translates $x \cdot \overline{K}_v$ for a basis of compact open subgroups $\overline{K}_v \subset \overline{G}(F_v)$. That is to say, f'_j is locally constant if and only if for all $x \in X_{\overline{K}_{S_0}}$ there is a compact open subgroup $\overline{K}_v$ depending on x such that f'_j is constant on $x \cdot \overline{K}_v$. However, by compactness of $X_{\overline{K}_{S_0}}$, we may choose the subgroup $\overline{K}_p$ so that it doesn't depend on x. Thus, f is continuous if and only if f'_j is smooth for all j.

The above process starts from an element of $\mathcal{C}^0_E(X_{\overline{K}_{S_0}})$ and produces an element of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$. Moreover the process is clearly reversible due to all the equivalences in the proof, which shows the desired equaility. □

4.2. **Completed Hecke Algebras.** For a fixed $\overline{K}_v \subset \overline{G}(F_v)$, there is a Hecke algebra $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. This is just the $\mathcal{O}_E$-subalgebra of $\mathrm{End}(H^0(X_{\overline{K}_{S_0}}/\overline{K}_v, E))$ generated by the Hecke operators T_w and S_w for $w \notin S \cup S_G$. If $\overline{K}'_v \subset \overline{K}_v$, then there is a natural surjection $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}'_v) \to \mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. Define the completed Hecke algebra

$\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}) = \varprojlim_{\overline{K}_v} \mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. Giving the algebras $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$ the ϖ_E-adic topology, $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ inherits the inverse limit topology from those topolgies.

It is a theorem that $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ is a semilocal ring; i.e. $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}) = \prod_{\mathfrak{m}} \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\mathfrak{m}}$ where the product runs over all maximal ideals $\mathfrak{m} \subset \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$. Well-known results about $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ imply that the maximal ideals of $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ are in correspondence with the finite set of $\overline{G}$-modular k_E-valued representations $\overline{\rho}$ of G_F, with a $\overline{G}$-modular lift ρ that is unramified outside of $S \cup S_G$, with ramification at primes in S_0 specified by the level $\overline{K}_{S_0}$ and with $\overline{\rho}|_{G_{F_w}}$ being an extension of an unramified character χ by $\chi\epsilon$ for all $w \in S_G$. However, it is possible that such a representation is decomposable when restricted to G_{F_w} and reduced mod ϖ_E even though it is indecomposable in $\mathcal{O}_E$. On the other hand, if $\overline{\rho}|_{G_{F_w}}$ is a nonzero extension of $\overline{\chi}$ by $\overline{\chi\epsilon}$, then the following lemma tells you that any (GL_2-)modular lift of $\overline{\rho}$ will be $\overline{G}$-modular.

Lemma 4.2.1. *Assume that $\#(k_w) \not\equiv \pm 1 \pmod{p}$, and let $\overline{\chi}$ be a mod ϖ_E character of G_{F_w}. If $\overline{\rho}_w : G_{F_w} \to GL_2(k_E)$ is a nontrivial extension of $\overline{\chi}$ by $\overline{\chi\epsilon}$, then any $\rho_w : G_{F_w} \to GL_2(\mathcal{O}_E)$ with $\rho_w \otimes_{\mathcal{O}_E} k_E = \overline{\rho}_w$ must be an extension of χ by $\chi\epsilon$ for some unramified character χ.*

Proof. Let ρ_w and $\overline{\rho}_w$ be as above. Notice that I_{F_w} must act on ρ_w through $\{\left(\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right) \pmod{\varpi_E}\}$, which is a pro-p group. Thus, I_{F_w} acts through the $\mathbb{Z}_p(1)$ factor in tame inertia. Let γ be a topological generator of $\mathbb{Z}_p(1)$. First, we need to show that γ does not act semisimply on ρ_w.

If γ acted semisimply on ρ_ℓ, then one must have two eigenspaces generated by vectors e_1 and e_2, and with eigenvalues ζ_1 and ζ_2. If Frob_w does not switch the eigenspaces, then the action of G_{F_w} on ρ_w and thus $\overline{\rho}_w$ must be through the abelian quotient of G_{F_w}. But this would imply that $\overline{\epsilon}$ is the trivial character and thus $\#(k_w) \equiv 1 \pmod{p}$. Thus, Frob_w must switch the eigenspaces.

Now, using the fact that $\mathrm{Frob}_\ell \gamma \mathrm{Frob}_\ell^{-1} = \gamma^\ell$, one gets that $\zeta_1^{\#(k_w)} e_1 = \gamma^{\#(k_w)} e_1 = \mathrm{Frob}_w \gamma \mathrm{Frob}_w^{-1} e_1 = \zeta_2 e_1$, and similarly $\zeta_2^{\#(k_w)} e_2 = \zeta_1 e_2$. Thus, $\zeta_1^{\#(k_w)^2 - 1} = 1$. Since ζ_1 is a p-power root of unity, one has $\#(k_w)^2 - 1 \equiv 1 \pmod p$, a contradiction.

Since γ does not act semisimply, ρ_ℓ must be an extension of χ by $\chi\epsilon$ for some character χ. Since $\overline{\chi}$ is unramified and $\#(k_w) \not\equiv 1 \pmod p$, $\mathcal{O}_{F_w}^\times$ has no quotients onto a p-group. Thus, χ must also be unramified, proving the lemma. $\square$

The above lemma is an incredibly simple case of "Jacquet-Langlands in Families," an analogue of Emerton and Helm's work on local Langlands in families for representations of quaternion algebras. This lemma forces that one has that the set of places S_G contains no places v with $\#(k_v) \equiv \pm 1 \pmod p$. From now on, we will assume this condition. The nonexistence of this work in full generality is the obstruction to letting there be level at the places in S_G.

With the above discussion in mind, assume that $\overline{\rho}$ is modular, absolutely irreducible, unramified outside of $S \cup S_G$, and has $\overline{\rho}|_{G_{F_w}} = \left(\begin{smallmatrix} \overline{\chi} & * \\ & \overline{\chi}\epsilon \end{smallmatrix}\right)$ with $*$ nonzero for all $w \in S_G$. If $\overline{\rho}$ corresponds to $\mathfrak{m}$, write $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0}) = \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\mathfrak{m}}$. There is a deformation $\rho(\overline{K}_{S_0})$ over $\mathbb{T}_{\overline{G},\overline{\rho}}$ such that $\mathrm{tr}(\mathrm{Frob}_w|_{\rho(\overline{K}_{S_0})}) = T_w$ and $\det(\mathrm{Frob}_w|_{\rho(\overline{K}_{S_0})}) = \#(k_w) S_w$. for all $w \notin S \cup S_G$. Since $\overline{\rho}$ is absolutely irreducible, there is a ring $R_{\overline{\rho}, S \cup S_G}$ parameterizing lifts of $\overline{\rho}$ unramified outside of S, together with a universal deformation $\rho^u / R_{\overline{\rho}, S \cup S_G}$. One has a map from $R_{\overline{\rho}, S\cup S_G} \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. This map is surjective, as $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$ is topologically generated by the T_ℓ and S_ℓs, which must be the image of $\mathrm{tr}(\mathrm{Frob}_\ell|_{\rho^u})$ and $\ell^{-1}\det(\mathrm{Frob}_\ell|_{\rho^u})$ respectively.

If $\overline{K}'_{S_0} \subset \overline{K}_{S_0}$, then one has a natural map from $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K'_{S_0}}) \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. Moreover, the map is surjective, as the following diagram commutes and all other maps are surjective:

$$\begin{array}{ccc} R_{\overline{\rho},S\cup S_G} & & \\ \downarrow & \searrow & \\ \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K'_{S_0}}) & \longrightarrow & \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0}). \end{array}$$

Any modular lift of $\overline{\rho}$ unramified outside of $S \cup S_G$ will have conductor away from $\{p\} \cup S_G$ dividing some integer $N_{\overline{\rho},S_0}$ by results of Carayol and Livné in [5] and [19]. This implies that if $\overline{K}_{S_0}$ is sufficeintly small (for example, having $\overline{K}_{S_0}$ contained in the principal congruence subgroup mod $N_{\overline{\rho},S_0}$ is sufficient), then the map $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K'_{S_0}}) \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$ is an isomorphism. Define $\mathbb{T}_{\overline{G},\overline{\rho},S} = \varprojlim_{\overline{K}_{S_0}} \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$, and notice that the transition maps are eventually isomorphisms. Thus, one has that $\mathbb{T}_{\overline{G},\overline{\rho},S}$ is a complete noetherian $\mathcal{O}_E$ algebra. Moreover, there is a surjection from $R_{\overline{\rho},S\cup S_G} \to \mathbb{T}_{\overline{G},\overline{\rho},S}$, which gives rise to a universal modular representation $\rho_S^m : G_F \to \mathrm{GL}_2(\mathbb{T}_{\overline{G},\overline{\rho},S})$. The characterizing property of ρ_S^m is that $T_w = \mathrm{tr}(\mathrm{Frob}_w|_{\rho_S^m})$ and $\#(k_w)S_w = \det(\mathrm{Frob}_w|_{\rho_S^m})$. We will say $\overline{K}_{S_0}$ is allowable if $\mathbb{T}_{\overline{G},\overline{\rho},S} \to \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is an isomorphism.

Since $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})$ is a $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$-module, one may localize at $\mathfrak{m}$ and obtain a $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$-module, denoted $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. Passing to the inverse limit over $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$, one gets a $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module denoted $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$. This should be thought of as the $\overline{\rho}$-part of $\hat{H}^0_{\mathcal{O}_E,\overline{G},S}$.

4.3. **Local-Global Compatability.** At this point, the discussion will focus on the $F = \mathbb{Q}$ case, as everything that will be said relies on the existence of a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\pi_S^m = B(\rho_S^m|_{G_{\mathbb{Q}_p}})$, the admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{T}_{\overline{G},\overline{\rho},S}$ given by the p-adic Langlands correspondence. We will let $\overline{\pi_S^m}$ be $\pi_S^m \otimes_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \mathbb{T}_{\overline{G},\overline{\rho},S}/\mathfrak{m}$, the mod p representation that π_S^m is a deformation of. Additionally, there is a representation $\pi_{S_0}(\rho_S^m)$ of $\overline{G}_{S_0}$, which is a smooth coadmissible representation of $\overline{G}_{S_0}$

over $\mathbb{T}_{\overline{G},\overline{\rho},S}$ (see Appendix [?] for an overview of the theory of local Langlands in families). Recall that $\pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m) := \varinjlim_{\overline{K}_{S_0}} \pi_S^m \hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m)^{\overline{K}_{S_0}}$.

Theorem 4.3.1. *There is an isomorphism of $\mathbb{T}_{\overline{G},\overline{\rho},S}$-modules with an action of $D^\times \times \overline{G}_{S_0}$:*

$$\pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m) \tilde{\to} \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}.$$

4.4. **Proof of Theorem 4.3.1.** Let $X = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p),\mathbb{T}_{\overline{G},\overline{\rho},S}}(\pi_S^m, \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S})$. There is an action of $\overline{G}_{S_0}$ on X. There is a natural evaluation map $\mathrm{ev}_X : \pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} X \to \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$, and if Y is a submodule of X then one may consider ev_Y as well. These maps will be the object of study for the rest of the section.

Say that a maximal prime $\mathfrak{p} \subset \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]$ is weakly allowable if $\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}$ is crystalline with distinct Hodge-Tate weights, and allowable if it is irreducible as well. While a direct proof that the allowable points are Zariski-dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$ would be desirable, it is easier to show that the closure of the allowable points contains the weakly allowable points and the weakly allowable points are dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

We will first determine the structure of the locally algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$. Let $\overline{W}$ be an irreducible algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\overline{K}_p$ be a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then the $\overline{W}, \overline{K}_p$-algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$ are the image of the evaluation map

$$\overline{W} \otimes_E \mathrm{Hom}_{\overline{K}_p}(\overline{W}, \hat{H}^0_{E,\overline{G},\overline{\rho},S}) \to \hat{H}^0_{E,\overline{G},\overline{\rho},S}.$$

The $\overline{W}$-algebraic vectors are the union over all $\overline{K}_p$ of the $\overline{W}, \overline{K}_p$-algebraic vectors and the $\overline{K}_p$-algebraic vectors are the union over all $\overline{W}$ of the $\overline{W}, \overline{K}_p$-algebraic vectors.

Lemma 4.4.1. *For $\overline{K}_{S_0}$ sufficiently small, one has the following:*

(1) $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}^{GL_2(\mathbb{Z}_p)-alg} = \bigoplus_{\mathfrak{p}} \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}[\mathfrak{p}]^{GL_2(\mathbb{Z}_p)-alg}$ with the sum taken over all weakly allowable primes $\mathfrak{p}$.

(2) The $GL_2(\mathbb{Z}_p)$-algebraic vectors are dense in $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$.

Proof. Let $\overline{W}$ be an algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. There is a local system $\mathcal{V}_{\overline{W}}$ over $X_{\overline{K}_{S_0}}/\overline{K}_p$ whose sections are $H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{V}_{\overline{W}}) = \{f : X_{\overline{K}_{S_0}} \to \overline{W} | f(gk) = k^{-1} \cdot f(g) \, \forall k \in \overline{K}_p\}$. Let $H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\overline{W}}) = \varinjlim_{\overline{K}_p} H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{V}_{\overline{W}})$. The space has a decomposition $H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\overline{W}}) = \bigoplus_\pi \pi_p \otimes (\pi_f^p)^{\overline{K}_{S_0} \overline{K}_0^S}$, where the sum is taken over all automorphic representations π of $\overline{G}(\mathbb{A})$ with $\pi_\infty = \check{\overline{W}}$ (this makes sense after choosing an embedding $E \hookrightarrow \mathbb{C}$).

Applying Corolloray 2.2.25 of [14], one sees that $\mathrm{Hom}_{\mathfrak{gl}_2}(\overline{W}, \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}))^{la} = H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\check{\overline{W}}})$. Thus, one has $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^{alg} = \oplus_{\overline{W},\pi} \overline{W} \otimes \pi_p \otimes (\pi_f^p)^{\overline{K}_{S_0} \overline{K}_0^S}$, the sum being taken over all algebraic representations $\overline{W}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ and automorphic representations π of $\overline{G}(\mathbb{A})$ with $\pi_\infty = \overline{W}$. Comparing Hecke actions, one sees that if π corresponds to a representation $\rho(\mathfrak{p})$, then the image of the π-part of the above decomposition must be $\mathfrak{p}$-torsion. The $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors arise when π_p has a $\mathrm{GL}_2(\mathbb{Z}_p)$ fixed vector. That happens only when $\mathfrak{p}$ is crystalline with distinct Hodge-Tate weights, namely when $\mathfrak{p}$ is allowable. This shows part 1 of the lemma.

A few words about the results in [14]. The paper works in a large amount of generality. For any reductive group over a number field G/F and place $v|p$ of F, there is a locally symmetric space $Y(K^v K_v) = G(\mathbb{Q}) \backslash G(\mathbb{A}_F)/K_\infty^\circ K^v K_v$. There are local systems $\mathcal{V}_W$ for any algebraic representation W of $G(F_v)$, and then the paper constructs a spectral sequence with $E_2^{i,j}$-page given by $\mathrm{Ext}^i_{\mathfrak{g}}(\check{W}, \hat{H}^i_E(Y(K^v))^{la}) \Rightarrow H^{i+j}(\mathcal{V}_W)$. Intuitively, this should be thought of as coming from the natural evaluation map $W \otimes \check{W} \to E$, which will give rise to a map from $H^i(\mathcal{V}_W, K^v) \otimes W \to \hat{H}^i_E(Y(K^v))$. After using adjoint functors, this map is the edge map of the spectral sequence mentioned before. Finally, the spectral sequence degenerates instantly in this case, as there are no H^is for $i \geq 1$.

For part 2, we claim that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is an injective $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module. To that end, let M be a smooth finitely generated $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$ module. Define $M^\vee = \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, \mathcal{O}_E/\varpi_E^s)$, the Pontrjagin dual of M. If $\overline{K}_{S_0}$ is sufficiently small, there is a local system $\mathcal{M}^\vee$ over $X_{\overline{K}_{S_0}}/\overline{K}_p$. Then one has $\mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[\overline{K}_p]}(M, \hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})) = H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{M}^\vee)$. Thus, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of smooth finitely generated $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-modules, there is a short exact sequence of local systems $0 \to \mathcal{M}_3^\vee \to \mathcal{M}_2^\vee \to \mathcal{M}_1^\vee \to 0$. This gives rise to a long exact sequence on cohomology, but there are no H^1s. Thus, the functor $M \to \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[\overline{K}_p]}(M, \hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0}))$ is exact, so $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is an injective $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module.

The injectivity of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ as an $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module is equivalent to the projectivity of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee$ as an $\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]]$-module. It is well known that $\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]]$ is a (noncommutative) noetherian local ring if $\overline{K}_p$ is a pro-p group. Thus, one has that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee = (\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]])^{r_s}$ for an r_s that depends on s. Consequently, $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0}) \cong \mathcal{C}^0(\overline{K}_p, \mathcal{O}_E/\varpi_E^s)^{r_s}$. Tensoring both sides with k_E, one gets $\hat{H}^0_{k_E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}^0(\overline{K}_p, k_E)^{r_s}$. The first side is visibly independent of s and so the second side must be too. Letting $r = r_s$, one gets that $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}(\overline{K}_p, \mathcal{O}_E)^r$, i.e. $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})^\vee = \mathcal{O}_E[[\overline{K}_p]]^r$, a free $\mathcal{O}_E[[\overline{K}_p]]$-module.

Now, choose $\overline{K}_p$ to be a sufficiently small (pro-p and normal works) subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. The natural map between $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]])}(\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee, *)$ and $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\overline{K}_p]])}(\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee, *)^{\mathrm{GL}_2(\mathbb{Z}_p)/\overline{K}_p}$ is an isomorphism. However, the second functor is exact, as $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee$ is free as an $E \otimes (\mathcal{O}_E[[\overline{K}_p]])$-module and taking invariants of a finite group is an exact functor in characteristic 0. Thus, $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee$ is projective as an $E \otimes (\mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]])$-module. Since projectivity is equivalent to being a summand of a free module, and taking duals, one gets that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is a summand of $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)^t$ for some t. Since $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is a summand of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$, to prove part 2, it is sufficient to show that the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors are dense in $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)$. However, the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors in $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)$ are the polynomial functions, and the theory of

Mahler expansions shows that polynomials are dense in the space of continuous functions. Thus, part 2 holds. ☐

Corollary 4.4.2. *The allowable points are Zariski-dense in* $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

Proof. Let $t \in \cap_{\mathfrak{p}}\mathfrak{p}$, the intersection being taken over all weakly allowable points. Then one has that $t \cdot v = 0$ for all $v \in (\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}})^{\mathrm{GL}_2(\mathbb{Z}_p)-alg}$. Thus, by the above lemma, one has $t \cdot \hat{H}^0_{E,G}(\overline{K}_{S_0})_{\overline{\rho}} = 0$. If $\overline{K}_{S_0}$ is sufficiently small, one has that $\mathbb{T}_{\overline{G},\overline{\rho},S} = \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ and since the action of $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ on $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is faithful, $t = 0$. This implies that the weakly allowable points are Zariski-dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

Proposition 5.4.9 in [15] shows that the set of allowable points are dense in the set of weakly allowable points, completing the proof of the corollary. ☐

For a maximal prime $\mathfrak{p} \subset \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]$, define $M(\mathfrak{p})$ to be the colsure of $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$ in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]$. If $\mathfrak{p}$ is an allowable prime, then one has that

$$\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg} = BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}),$$

where $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$ is the locally algebriac representation defined in [4]. The seminal result of Berger and Breuil in [1] shows that for allowable points, $\pi^m_S(\mathfrak{p})$ is the universal unitary completion of $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$. Thus, taking the $M(\mathfrak{p})^{\overline{K}_{S_0}}$, one gets a complete Banach space that contains $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}}) \otimes \left(\bigotimes_{\ell \in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})\right)^{\overline{K}_{S_0}}$ as a dense subspace. After taking the limit over K_{S_0}, one gets

$$M(\mathfrak{p}) = \pi^m_S(\mathfrak{p}) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}).$$

At this point, We are in a position to compute $(X \otimes E)[\mathfrak{p}]$. Using the fact that $\pi_S^m(\mathfrak{p})$ is the universal unitary completion of $BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$, one gets

$$\begin{aligned}(X \otimes E)[\mathfrak{p}] &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_S^m(\mathfrak{p}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_S^m(\mathfrak{p}), M(\mathfrak{p})) \\ &= \bigotimes_{\ell \in S_0} \pi_{LL}(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})\end{aligned}$$

From the above chain, it is visible that $\mathrm{ev}_X : \pi_S^m(\mathfrak{p}) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}) \to M(\mathfrak{p})$ is an isomoprphism.

Recall the following proposition from [15]:

Proposition 4.4.3. *Let Y be a saturated coadmissible $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X. Then the following are equivlant:*

(1) Y is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.

(2) For all allowable $\overline{K}_{S_0}$, $Y^{\overline{K}_{S_0}}$ is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.

(3) For all allowable primes $\mathfrak{p}$, the inclusion $Y[\mathfrak{p}] \hookrightarrow X[\mathfrak{p}]$ is an isomorphism.

(4) For all allowable primes $\mathfrak{p}$ and allowable levels $\overline{K}_{S_0}$, the inclusion $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \hookrightarrow X^{\overline{K}_{S_0}}[\mathfrak{p}]$ is an isomorphism.

(5) $Y \supset X_{ctf}$, where X_{ctf} is the maximal cotorsion free submodule of X as in definition C.39 in [15].

Proof. The proof will be recalled here as well. It is immediate that 2) $\Rightarrow$ 1) and that 3) $\Leftrightarrow$ 4). Thus, all that is needed for the equivalence of 1) through 4) is 1) $\Rightarrow$ 3) and 4) $\Rightarrow$ 2).

The above description of $X[\mathfrak{p}]$ shows that $X[\mathfrak{p}]$ is an irreducible $\overline{G}_{S_0}$-representation for $\mathfrak{p}$ an allowable prime. Thus, $Y[\mathfrak{p}] \neq 0$ if and only if $Y[\mathfrak{p}] \hookrightarrow X[\mathfrak{p}]$ is an isomorphism. But Proposition C.36 of [15] shows that, if Y is faithful, then $Y[\mathfrak{p}] \neq 0$. Thus, 1) $\Rightarrow$ 3).

Similarly, if $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \hookrightarrow X^{\overline{K}_{S_0}}[\mathfrak{p}]$ is an isomorphimsm for allowable levels $\overline{K}_{S_0}$ and allowable primes $\mathfrak{p}$, then one has that $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \neq 0$ for all allowable primes. Proposition C.22 of [15] implies that the allowable primes are in the cosupport of $Y^{\overline{K}_{S_0}}$ and thus by Zariski density of allowable primes, one has that the cosupport of $Y^{\overline{K}_{S_0}}$ is (0), i.e. Y is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.

The final step is to show that 5) is equivalent to any of the other parts. To that end, it is useful to show that X_{ctf} is the unique saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X. Proposition C.40 of [15] shows that, since X is a saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-module, so is X_{ctf}. Letting Y be a saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X, one has that the image of Y in X lies in X_{ctf}, and by 3), $Y[\mathfrak{p}] \hookrightarrow X_{ctf}[\mathfrak{p}]$ is an isomorphism. Proposition C.41 in [15] shows that $Y \hookrightarrow X_{ctf}$ is an isomorphism.

Thus, if Y is any saturated faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X, then Y_{ctf} is faithful as well. Since Y_{ctf} is cotorsion-free, one must have that $Y_{ctf} = X_{ctf}$ and thus $Y \supset X_{ctf}$. This shows that 1) $\Rightarrow$ 5). Conversely, if $Y \supset X_{ctf}$, then Y contains a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module and thus must be faithful itself, showing 5) $\Rightarrow$ 1). This completes the proof of the proposition. □

Proposition 4.4.4. *Let Y be a saturated submodule of X that satisfies the equivalent conditions of the above proposition. Then one has that* $\mathrm{ev}_{Y,E}$ *is surjective.*

Proof. We will show that $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E} : \pi_S^m \otimes Y^{K_{S_0}} \to \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is surjective for any allowable $\overline{K}_{S_0}$. This implies the result after taking the limit over all $\overline{K}_{S_0}$.

If $\mathfrak{p}$ is a weakly allowable prime, then one has that $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$ is irreducible as a $\mathrm{GL}_2(\mathbb{Q}_p) \times \overline{G}_{S_0}$-representation. Thus, in order to show that $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains

$\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$, it is necessary and sufficent to show that it contains one locally algebraic vector. However, one has that Y is faithful as a $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module, and so $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains a Jordan-Hölder factor of $\pi^m_S(\mathfrak{p})$. If $\rho^m_S(\mathfrak{p})$ is irreducible, then so is $\pi^m_S(\mathfrak{p})$ and remarks made above show that $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains locally algebraic vectors in the $\mathfrak{p}$-torsion. Otherwise, the other possible Jordan-Hölder factors are either a principal series associated to locally algebraic characters, a twist of the Steinberg by a locally algebraic character, or a locally algebraic character. "By hand" calculations on all three of the possibilities show that they all have locally algebraic vectors, and so the image of $\mathrm{ev}_{Y,E}$ contains one (hence all) of the locally algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]$. Taking $\overline{K}_{S_0}$-invariants, one gets that the image of $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E}$ contains all of the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors.

Thus, one has that the image of $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E}$ contains a dense subspace of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$. Lemma 3.1.16 in [15] implies that $\pi^m_S\hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}}Y^{\overline{K}_{S_0}}$ is an admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, the finiteness of the class group of $\overline{G}$ implies that $H^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is admissible. Proposition 3.1.3 of Emerton shows that the image of $\mathrm{ev}_{Y,E}$ is thus closed, and so must be everything. □

Lemma 4.4.5. *The following are equivalent:*

(1) ev_Y *is an isomorphism.*

(2) Y *is a faithful* $\mathbb{T}_{\overline{G},\overline{\rho},S}$*-module, and* $\mathrm{ev}_Y(\mathfrak{m}) : (Y/\varpi_E)[\mathfrak{m}] \otimes_{k_E} \overline{\pi^m_S} \to \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$ *is injective.*

If either of these conditions holds, then $Y = X_{ctf}$.

Proof. Assume first that ev_Y is an isomorphism. Because $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$ is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module, Y must be as well. Moreover, since ev_Y is an isomorphism, the associated map mod ϖ_E is injective and this remains true when passing to $\mathfrak{m}$ torsion. Thus, 1) $\Rightarrow$ 2).

If $\mathrm{ev}_Y(\mathfrak{m})$ is injective, the lemma C.46 of [15] implies that ev_Y is injective as well, with saturated image. Lemma C.52 of [15] implies that Y must necessarily be saturated in X. Since Y is faithful, Y satisfies the conditions of Proposition 4.4.3, and thus one has that $\mathrm{ev}_{Y,E}$ is surjective. But, as before, one has that the image of ev_Y is saturated, so $\mathrm{ev}_{Y,E}$ is surjective if and only if ev_Y is. Thus, ev_Y is an isomorphism, showing 2) $\Rightarrow$ 1).

Now assume that ev_Y is an isomorphism. Since Y is faithful, Proposition 4.4.3 shows that $Y \supset X_{ctf}$. Additionally, one has that $X_{ctf}/\varpi_E X_{ctf} \hookrightarrow Y/\varpi_E Y$ as X_{ctf} is saturated in X and thus Y. This remains injective when passing to $\mathfrak{m}$ torsion, and so one has that X_{ctf} satisfies the conditions of the lemma. Consequently one has that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism and so since $\mathrm{ev}_{X_{ctf}} = \mathrm{ev}_Y \circ \iota_{X_{ctf},Y}$, one has that $\iota_{X_{ctf},Y}$ is an isomorphism as well. That is, $Y = X_{ctf}$. □

A key point to make in the above lemma is that it does not prove that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism unconditionally. It only proves that if there is a submodule $Y \subset X$ such that ev_Y is an isomorphism, then $Y = X_{ctf}$. Indeed, the proof that X_{ctf} satifies condition 2) in Lemma 4.4.5 needs that there is a Y such that Y does.

Now, we will recall another key result in [15]. Recall that a Serre weight is a representation of $\mathrm{GL}_2(\mathbb{F}_p)$ over k_E. Such a weight is of the form $\mathrm{Sym}^a(\mathrm{Std}) \otimes \det^b$, with $0 \leq a \leq p-1$ and $0 \leq b \leq p-2$. If V is a Serre weight, then V is a global Serre weight for ρ if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \hat{H}^1_{E,\overline{G},\overline{\rho},S}) \neq 0$. The set of such weights is denoted $W^{gl}(\overline{\rho})$. A weight V of the form $\mathrm{Sym}^a(\mathrm{Std}) \otimes \det^b$ in $W^{gl}(\overline{\rho})$ is called good if either $a < p-1$ or $a = p-1$ and det^b is not in $W^{gl}(\overline{\rho})$. Notice that if V is not good, then necessarily det^b is a global Serre weight for $\overline{\rho}$ and that is a good Serre weight, so $W^{gl}(\overline{\rho})$ contains a good Serre weight.

Theorem 4.4.6. *(1)* $W^{gl}(\overline{\rho}) \subset W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$.

(2) *If* $V \in W^{gl}(\overline{\rho})$ *is a good weight, there is an isomorphism of* $\mathcal{H}(V)$*-modules*

$$F_{S_0}\left(soc_{\mathcal{H}(V)}\left(\mathrm{Hom}_{k_E[GL_2(\mathbb{Z}_p)]}(V, H^0_{k_E,\overline{G},\overline{\rho},S})[\mathfrak{m}]\right)\right) \cong soc_{\mathcal{H}(V)} m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}}).$$

(3) *For any* $V \in W^{gl}(\overline{\rho})$*, the* $\overline{G}_{S_0}$*-representation* $\mathrm{Hom}_{k[GL_2(\mathbb{Z}_p)]}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ *is generic.*

Part one of this theorem is one direction in the weight part of Serre's conjecture, part two is a mod p multiplicity one result, and part three is Ihara's lemma.

Corollary 4.4.7. $\mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ *is a generic* $\overline{G}_{S_0}$ *representation.*

Proof. The proof breaks up into three cases: $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension of χ_1 by χ_2 with $\chi_1\chi_2^{-1} \neq \overline{\epsilon}$, or $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension of χ by $\chi\overline{\epsilon}^{-1}$.

In the first case, one then has that $\overline{\pi^m_S}$ is irreducible. Let V be a weight in $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\overline{\pi^m_S})$, chosen to be one-dimensional if possible. Since $\overline{\pi^m_S}$ is irreducible, V generates $\overline{\pi^m_S}$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$ representation, and so the natural restriction map $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is injective. If the target is nonzero, then V is in $W^{gl}(\overline{\rho})$ and is good, so part three of the above theorem shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic and thus so is $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$. If the target is zero, then so is the source, and thus $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is trivially generic.

In the second case, one has that $\overline{\pi^m_S}$ is an extension of irreducible representations $0 \to \overline{\pi}_1 \to \overline{\pi^m_S} \to \overline{\pi}_2 \to 0$. Letting V_i be a weight for $\overline{\pi}_i$, the exact same argument as above shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_i, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic. Thus, since there is an exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_2, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$$
$$\to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_1, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$$

which shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ has a quotient by a generic representation that lies inside a generic representation, and thus is generic.

In the final case, one has that there is a short exact sequence as above $0 \to \overline{\pi}_1 \to \overline{\pi_S^m} \to \overline{\pi}_2 \to 0$, where $\overline{\pi}_2 = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi\overline{\epsilon}^{-1} \boxtimes \chi\overline{\epsilon})$ and $\overline{\pi}_1$ now lies in a short exact sequence $0 \to (\chi \circ \det) \otimes St \to \overline{\pi}_1 \to \chi \circ \det \to 0$. A simple calculation shows that $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\overline{\pi}_2)$ has one weight that is not in $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$, and so one has that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_2, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$. Thus, the map $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_1, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is an injection.

If $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is trés ramifieé, then $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$ contains no one dimensional weights. Thus, $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\chi \circ \det, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$ and so $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi \circ \det, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$. Conseqently, we get that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ embeds into $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi \circ \det) \otimes St, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$. Moreover, any weight for $(\chi \circ \det) \otimes St$ is good because there are no one dimensional weights in $W^{gl}(\overline{\rho})$, and the argument used above shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic.

If, on the other hand, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is peu ramefieé, then the natural surjection $\overline{\pi}_1 \to \chi \circ \det$ admits a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant section whose image generates $\overline{\pi}_1$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. Thus, $\chi \circ \det$ is a weight for $\overline{\rho}|_{G_{\mathbb{Q}_p}}$, which must be good. Thus, arguing as above, one sees that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic. □

Lemma 4.4.8. *If* $\mathrm{ev}_{X_{ctf}}$ *is an isomorphism, then one has that* $X_{ctf} = \pi_{S_0}(\rho_S^m)$

Proof. We will let $\mathcal{C}$ be the set of weakly allowable primes in $\mathbb{T}_{\overline{G},\overline{\rho},S}$. Then, with this choice of $\mathcal{C}$, one needs to show that X_{ctf} satisfies the conditions of Theorem A.4.1. This will show that there is an isomorphism $X_{ctf} \cong \pi_{S_0}(\rho_S^m)$.

Because X_{ctf} is saturated in X, there is a chain of embeddings

$$(X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] \hookrightarrow (X/\varpi_E X)[\mathfrak{m}] \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]).$$

Since the last term in the chain is generic as a $\overline{G}_{S_0}$ representation by Corollary 4.4.7, so is the first. This shows the first condition in Theorem A.4.1.

By the assumption, one has that $\mathrm{ev}_{X_{ctf}}(\mathfrak{m})$ is an embedding of $(X_{ctf}/\varpi_E X_{ctf}) \otimes_{k_E} \overline{\pi_S^m} \hookrightarrow H^0_{k_E,\overline{G},\overline{\rho},S}$. Letting $V \in W^{gl}(\overline{\rho})$ be a good weight, one has an isomorphism $m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}}) \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m})$. Then there is the following chain of maps which are all embeddings (indeed, all but one are isomorphisms):

$$\begin{aligned}
&\mathrm{soc}_{\mathcal{H}(V)}(m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}})) \otimes_{k_E} F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]) \\
&= \mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m})) \otimes_{k_E} F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]) \\
&= F_{S_0}(\mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m} \otimes_{k_E} (X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]))) \\
&\hookrightarrow F_{S_0}(\mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]))) \\
&\cong \mathrm{soc}_{\mathcal{H}(V)}(m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}})).
\end{aligned}$$

The last isomorphism is from part two of Theorem 4.4.6. This implies that the space $F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}])$ is at most one-dimensional. This shows the second condition in TheoremLocalLanglandsInFamilies.

By the discussion about allowable primes $\mathfrak{p}$, one has that $X_{ctf}[\mathfrak{p}] \otimes_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]/\mathfrak{p} = X_{ctf}[\mathfrak{p}] \otimes_{\mathcal{O}_E} E = \otimes_{\ell \in S_0} \pi_{LL}(\rho(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})$, and then [17] shows that that representation is generic. This shows the first half of condition 3 in Theorem A.4.1. All that is left is to show that the closure of the saturation of $\Sigma_{\mathfrak{p} \in \mathcal{C}} X_{ctf}[\mathfrak{p}]$ is all of X_{ctf}.

If one lets Y be the closure of the saturation of $\Sigma_{\mathfrak{p} \in \mathcal{C}} X_{ctf}[\mathfrak{p}]$, then one has that $E \otimes_{\mathcal{O}_E} Y[\mathfrak{p}] = E \otimes_{\mathcal{O}_E} X[\mathfrak{p}]$ for all allowable primes $\mathfrak{p}$, and thus $Y \supset X_{ctf}$. But one also has that Y is the closure of a subspace of X_{ctf}, and thus $Y = X_{ctf}$, showing part c of 2. Thus, one has that $X_{ctf} = \pi_{S_0}(\rho_S^m)$. □

Lemma 4.4.9. $\mathrm{ev}_{X_{ctf}}(\mathfrak{m})$ *is injective.*

Proof. Recall that $\mathrm{ev}_{X_{ctf}} : \overline{\pi_S^m} \otimes_{k_E} (X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$. Since one has that $(X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$, Lemma 6.4.15 of [15] shows that it is sufficient to show that any non-zero map $\overline{\pi_S^m} \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$ is injective. Again, the proof breaks up into three cases: $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is the extension of χ_1 by χ_2 with $\chi_1\chi_2^{-1} \neq \overline{\epsilon}$, or $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is the extension of χ by $\chi\overline{\epsilon}^{-1}$.

In the first case, one has that $\overline{\pi_S^m}$ is irreducible, and so any non-zero $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map must be injective.

In the second case, one has that $\overline{\pi_S^m}$ is an extension of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2\overline{\epsilon})$ by $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1\overline{\epsilon})$. $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$ consists of a single Serre weight, and that weight corresponds to the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2\overline{\epsilon})$. Since the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1\overline{\epsilon})$ is not the same, one has that there is no non-zero map from $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2\boxtimes\chi_1\overline{\epsilon}) \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$. Thus, any non-zero map from $\overline{\pi_S^m} \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$ must be non-zero on $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2\overline{\epsilon})$ and thus is injective.

In the final case, there is a filtration $0 \subset \overline{\pi}_1 \subset \overline{\pi}_2 \subset \overline{\pi_S^m}$ with $\overline{\pi}_1 = (\chi \circ \det) \otimes_{k_E} St$, $\overline{\pi}_2/\overline{\pi}_1 = \chi \circ \det$, and $\overline{\pi_S^m}/\overline{\pi_2} = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi\overline{\epsilon}^{-1} \boxtimes \chi\overline{\epsilon})$. The same Serre weight considerations as above show that there is no non-zero map from $\overline{\pi_S^m}/\overline{\pi}_2 \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$. Additionally, Ihara's lemma guarentees that there is no non-zero map from $\overline{\pi}_2/\overline{\pi}_1 \to H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$, as the image would be one dimensional. Thus, any non-zero map must have trivial kernel, and thus is injective. $\square$

Proof of Theorem 4.3.1. Lemmas 4.4.5 and 4.4.9 show that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism. Lemma 4.4.8 shows that, if $\mathrm{ev}_{X_{ctf}}$ is an isomorphism, then $X_{ctf} = \pi_{S_0}(\rho_S^m)$. Thus, $\mathrm{ev}_{X_{ctf}}$ provides an isomorphism between $\pi_S^m \hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m)$ and $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$, showing the theorem. $\square$

5. Analysis of the Completed H^1 for G

The aim of this section is to understand the completed cohomology for G. The first main theorem of the paper will be shown here.

5.1. **Completed Cohomology and Completed Hecke Algebras.** As before, let S_0 be a finite set of places of F. Assume that $v \notin S_0$ and that any place w where $G(F_w) \neq \mathrm{GL}_2(F_w)$ is not in S_0. Then let $S = S \cup \{p\}$ and $S_G = \{w | G(F_w) \neq \mathrm{GL}_2(F_w)\}$, and choose a maximal compact subgroup $K_0^S \subset G(\mathbb{A}_{F,f}^S)$. Let $G_{S_0} = \prod_{w \in S_0} G(F_w)$. Then recall the following set of definitions:

Definition 5.1.1. *If $K_{S_0} \subset G_{S_0}$ is a compact open subgroup, then define*

$$\hat{H}^i_{\mathcal{O}_E,G}(K_{S_0}) = \varprojlim_s \varinjlim_{K_v} H^i_{\acute{e}t}(Sh_{K_v K_{S_0} K_0^S, \overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s).$$

Additionally, let $\hat{H}^i_{E,G}(K_{S_0}) = \hat{H}^i_{\mathcal{O}_E,G}(K_{S_0}) \otimes_{\mathcal{O}_E} E$ and $\hat{H}^i_{*,G,S} = \varinjlim_{K_{S_0}} \hat{H}^i_{*,G}(K_{S_0})$.

$\hat{H}^1_{E,G}(K_{S_0})$ is an admissible unitary representation of $D^\times_{F_v}$. Additionally, there is an action of G_L (or in the case of $\mathbb{Q}$ one gets an action of $G_{\mathbb{Q}}$) on $\hat{H}^1_{*,G}(K_{S_0})$ that commutes with the action of $D^\times_{F_v}$. Moreover, G_{S_0} acts smoothly on $H^1_{E,G,S}$. The last action that needs to be discussed is that of a Hecke algebra. Let $\mathbb{T}_G(K_v K_{S_0})$ be the $\mathcal{O}_E$ subalgebra of $\mathrm{End}(H^1_{\acute{e}t}(Sh_{K_v K_{S_0} K_0^S, \overline{F}}, E))$ generated by the operators T_w and S_w for $w \notin S \cup S_G$. As before, there is a surjection $\mathbb{T}_G(K'_v K_{S_0}) \to \mathbb{T}_G(K_v K_{S_0})$ if $K'_v \subset K_v$. As before, let $\mathbb{T}_G(K_{S_0}) = \varprojlim_{K_v} \mathbb{T}_G(K_v K_{S_0})$, the completed Hecke algebra for K_{S_0}. If $\overline{\rho}$ is a G-modular, absolutely irreducible 2-dimensional k_E-valued representation of G_L that is unramified outside of $S \cup S_G$, then there is a maximal ideal $\mathfrak{m} \subset \mathbb{T}_G(K_{S_0})$ such that $T_w \equiv tr(\mathrm{Frob}_w|_{\overline{\rho}}) \pmod{\mathfrak{m}}$ and $S_w \equiv \#(k_w)^{-1}\det(\mathrm{Frob}_w|_{\overline{\rho}})$. We will let $\mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ be $\mathbb{T}_G(K_{S_0})_{\mathfrak{m}}$. The same arguemnt from section 4.2 shows that there is a compact open subgroup $K_{S_0} \subset G_{S_0}$ such that if $K'_{S_0} \subset K_{S_0}$, then the map from $\mathbb{T}_{G,\overline{\rho}}(K'_{S_0}) \to \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ is an isomorphism. Let $\mathbb{T}_{G,\overline{\rho},S}$ be $\varprojlim_{K_{S_0}} \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ and say that K_{S_0} is an allowable level if $\mathbb{T}_{G,\overline{\rho},S} = \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$.

Recall from section 4.2 that there is also a Hecke algebra $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$. Since the two Hecke algebras are determined by the corresponding Hecke eigensystems (that is, maps from $\mathbb{T}$ to finite extensions of E) and any Hecke eigensystem for G necessarily gives one for $\overline{G}$, there is a surjection from $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}) \to \mathbb{T}_G(K_{S_0})$. Additionally, this map commutes with the localization mentioned above, so there is a surjection $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0}) \to \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$, and this extends to a surjection $\mathbb{T}_{\overline{G},\overline{\rho},S} \to \mathbb{T}_{G,\overline{\rho},S}$.

5.2. **Statement of Results.** Let $\mathfrak{p}$ be a maximal ideal in $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$. Then, by looking at $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}]$, one gets a representation of $\mathrm{GL}_2(F_v)$ that depends only on $\mathfrak{p}$ and K_{S_0} which will be denoted $\pi_{\mathfrak{p},K_{S_0}}$. Conjecturally, this representation should be of the form $\pi_{\mathfrak{p},K_{S_0}} = \pi(\rho^m_S(\mathfrak{p})|_{G_{F_v}}) \otimes (\bigotimes_{w\in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{F_w}}))^{\overline{K}_{S_0}}$ with $\pi(\rho^m_S(\mathfrak{p})|_{G_{F_v}})$ being a unitary representation of $\mathrm{GL}_2(F_v)$ that depends only on $\rho^m_S(\mathfrak{p})|_{G_{L_{v_1}}}$, but the incomplete knowledge of the p-adic Langlands program prevents such a decomposition from being known. Since $\mathbb{T}_G(K_{S_0})$ is a $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$-algebra, one may also talk about $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}]$, which is a representation of $D^\times_{F_v} \times G_L$. The first main result is the following:

Theorem 5.2.1. *The space $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}]$ depends only on $\pi_{\mathfrak{p},K_{S_0}}$ as a $D^\times_{F_v} \times G_{F_v}$-representation.*

Since the main difficulty in getting the optimal result in Theorem 5.2.1 is the lack of a p-adic Langlands correspondence in generality, one would hope that there is a stronger theorem for the case when $F = \mathbb{Q}$ and $F_v = \mathbb{Q}_p$. Indeed, there is, which is given by the following:

Theorem 5.2.2. *If ρ is a two-dimensional promodular representation of $G_{\mathbb{Q}}$, unramified away from S, and such that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is indecomposable and not of the form $\begin{pmatrix} \chi & * \\ & \chi \end{pmatrix}$ nor $\begin{pmatrix} \chi & * \\ & \chi\overline{\epsilon} \end{pmatrix}$, $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ is of the form $\begin{pmatrix} \chi & * \\ & \chi\overline{\epsilon} \end{pmatrix}$ for all $\ell \in S_G$, and ρ is unramified away from $S \cup S_G$, then there is a continuous unitary representation $J(\rho|_{G_{\mathbb{Q}_p}})$ of $D^\times_{\mathbb{Q}_p}$ that depends only on $\rho|_{G_{\mathbb{Q}_p}}$, and such that*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G,\overline{\rho},S}) = J(\rho|_{G_{\mathbb{Q}_p}}) \otimes (\bigotimes_{\ell\in S_0} \pi_{\ell\ell}(\rho|_{G_{\mathbb{Q}_\ell}})).$$

5.3. **Proofs of Main Theorems.** Recall that the Čerednik-Drinfel'd uniformization breaks up as $Sh_{K_v^n K_{S_0} K_0^S, \mathbb{C}_p} = \coprod_i \Gamma_i' \backslash \Sigma_{\mathbb{C}_p}^n$, with the Γ_i' being discrete cocompact subgroups of $\mathrm{SL}_2(F_v)$. We will drop the $\mathbb{C}_p$-subscript and will write Σ^n for $\Sigma_{\mathbb{C}_p}^n$ until otherwise noted. Letting $\pi_i : \Sigma^n \to \Gamma_i' \backslash \Sigma^n$ be the natural projection. Then there is a commutative diagram of functors:

$$\begin{array}{ccc} \Gamma_i' - \acute{e}tSh(\Sigma^n) & \xrightarrow{H^0(\Sigma^n, \cdot)} & \Gamma_i' - Mod \\ \uparrow \pi_i^* & & \downarrow (\cdot)^{\Gamma_i'} \\ \acute{e}tSh(\Gamma_i' \backslash \Sigma^n) & \xrightarrow{\Gamma_{\Gamma_i' \backslash \Sigma^n}} & Ab. \end{array}$$

If K_{S_0} is sufficiently small, then the action of Γ_i' is free on Σ^n and thus π^* is an equivalence of categories. Additionally, $\pi^* \left(\underline{\mathcal{O}_E / \varpi_E^s} \right) = \underline{\mathcal{O}_E / \varpi_E^s}$, so applying the Grothendieck-Leray spectral sequence, one gets $R^a(\cdot)^{\Gamma_i'} \left(R^b \Gamma_{\Sigma^n} \left(\underline{\mathcal{O}_E / \varpi_E^s} \right) \right) \Rightarrow R^{a+b} \Gamma_{\Gamma_i' \backslash \Sigma^n} \left(\underline{\mathcal{O}_E / \varpi_E^s} \right)$. Giving the functors their more common names, one gets $H^a(\Gamma_i', H^b_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \Rightarrow H^{a+b}_{\acute{e}t}(\Gamma_i' \backslash \Sigma^n, \mathcal{O}_E/\varpi_E^s)$. In low degree terms, there is an exact sequence

$$0 \to H^1(\Gamma_i', H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \to H^1_{\acute{e}t}(\Gamma_i' \backslash \Sigma^n, \mathcal{O}_E/\varpi_E^s) \to H^1_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)^{\Gamma_i'}$$
$$\to H^2(\Gamma_i', H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)).$$

The condition that Γ_i' is a discrete cocompact subgroup of $\mathrm{SL}_2(F_v)$ means that Γ_i' is an essentially free group. If one choose K_{S_0} to be sufficiently small, then one has that Γ_i' is a free group. Thus, for sufficiently small K_{S_0}, there is no $H^2(\Gamma_i', *)$ for any choice of $*$.

The other term that admits easy analysis is the $H^1(\Gamma_i', *)$ term. Since (again, if K_{S_0} is sufficiently small) Γ_i' acts freely on the tree $\mathcal{T}$ for $\mathrm{PGL}_2(F_v)$, one has that $H^1(\Gamma_i', *) = H^1(\Gamma_i' \backslash \mathcal{T}, *)$ for any choice of $*$. As a point of clarity, the cohomology on the right in that equality is simply Betti cohomology of a graph.

We will consider the space of harmonic cochains on the graphs $\Gamma_i'\backslash\mathcal{T}$. Since the graphs that will be considered are all $(p+1)$-regular and the coefficient groups for the cochains are all $\mathbb{Z}_p$-modules, the condition of harmonicity is insensitive to the standard normalization issues and the choice of whether or not to divide by the degree will not change whether a cochain is harmonic. Since the top dimensional simplex is 1-dimensional, there is only one condition for a cochain f being harmonic, namely that $\sum_{v'} f(\overrightarrow{vv'}) = 0$ for all verticies v, where the sum is taken over all verticies v' adjacent to v. If H is a $(p+1)$-regular graph and M is an abelian group, then let $\mathrm{Harm}^1(H, M)$ be the space of M-valued harmonic 1-cochains on H.

There is always a map from $\mathrm{Harm}^1(H, M) \to H^1(H, M)$ sending a cocycle to its image in H^1. If M is a $\mathbb{Q}$-vector space, then this map is an isomorphism. In more generality, this map may have kernel and cokernel, but there is an integer N depending only on H such that the kernel and cokernel are N-torsion. Thus, passing to the inverse limit over s, there is no kernel. Additionally, tensoring with E will kill the cokernel, so one has an isomorphism between $\left(\varprojlim_s \varinjlim_n H^1(\Gamma_i', H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s))\right) \otimes_{\mathcal{O}_E} E$

and $\left(\varprojlim_s \varinjlim_n \mathrm{Harm}^1(\Gamma_i'\backslash\mathcal{T}, H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s))\right) \otimes_{\mathcal{O}_E} E$. Now, one has that

$$\begin{aligned}
&\bigoplus_{\Gamma_i'} \left(\left(\varprojlim_s \varinjlim_n \mathrm{Harm}^1(\Gamma_i'\backslash\mathcal{T}, H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \right) \otimes_{\mathcal{O}_E} E \right) \\
&= \bigoplus_{\Gamma_i'} \left(\left(\varprojlim_s \varinjlim_n \mathrm{Harm}^1(\mathcal{T}, H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \right)^{\Gamma_i'} \otimes_{\mathcal{O}_E} E \right) \\
&= \bigoplus_{\Gamma_i'} \left(\left(\varprojlim_s \mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}^0(\mathcal{O}_F^\times, \mathcal{O}_E/\varpi_E^s) \right)^{\Gamma_i'} \otimes_{\mathcal{O}_E} E \right) \\
&= \bigoplus_{\Gamma_i} \left(\left(\varprojlim_s \mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}^0(F^\times, \mathcal{O}_E/\varpi_E^s) \right)^{\Gamma_i} \otimes_{\mathcal{O}_E} E \right) \\
&= \left(\left(\varprojlim_s \mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}^0(F^\times \times X_{\overline{K}_{S_0}}, \mathcal{O}_E/\varpi_E^s) \right)^{\mathrm{GL}_2(F)} \otimes_{\mathcal{O}_E} E \right) \\
(1) \quad &= \left(\left(\left(\varprojlim_s \mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \right) \otimes_{\mathcal{O}_E} E \right) \hat{\otimes}_E \mathcal{C}^0(F^\times \times X_{\overline{K}_{S_0}}, E) \right)^{\mathrm{GL}_2(F)}
\end{aligned}$$

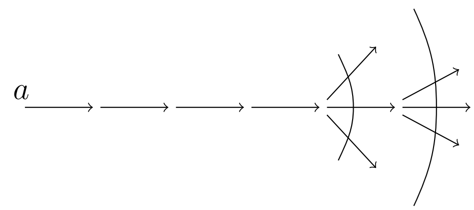
The equality between the first and second lines comes from the fact that a cochain being hamonic is a local condition, and the condition to descend from $\mathcal{T}$ to $\Gamma_i'\backslash\mathcal{T}$ is just being invariant under the action of Γ_i'.

The following lemma is useful for the analysis of (1).

Lemma 5.3.1. *There are no smooth harmonic $\mathcal{O}_E/\varpi_E^s$-valued 1-cochains on $\mathcal{T}$.*

Proof. It is useful to recall what all the adjectives in the above lemma mean. A 1-cochain on $\mathcal{T}$ is a function f on directed edges such that $f(\overrightarrow{ab}) = -f(\overrightarrow{ba})$. Being harmonic means $\sum_b f(\overrightarrow{ab}) = 0$ for all verticies a, where the sum is taken over all verticies b that are adjacent to a. Finally, smooth means that, for one (equivalently any) choice of vertex a to be the center of the tree, there is an integer n such that, if the distance from e to a is greater than n, then $f(e)$ depends only on the distance from e to a and the edge of distance n from a along the shortest path from a to e. The picture below shows the edges that must have the same value for $p = 3$ and

$n = 4$.



Now, let e be an edge that is distance $> n$ from a and oriented away from a (using the notation of the previous paragraph). Then one has that $f(e) = \sum_{e'} f(e')$, where the sum is over all edges adjacent to e and one unit farther from a. But smoothness implies that the values of $f(e')$ are constant. Thus, one has that $p | f(e)$. The same argument applies to $f(e')$ for e' one unit further from a, and thus one gets $p^2 | f(e)$. One can repeat this process until one has that $\varpi_E^s | f(e)$. But at that point, one must have $f(e) = 0$. Thus, $f(e) \equiv 0$ for all edges e distance greater than n from a. If e is distance exactly n from a, then for all e' adjacent to e and distance $n+1$ from a, one has that $f(e') = 0$. Thus, $f(e) = 0$ by harmonicity. This argument again repeats until one sees that $f \equiv 0$. $\square$

If there was a nonzero vector $v \in \left(\left(\left(\varprojlim_s \mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s)\right) \otimes_{\mathcal{O}_E} E\right) \hat{\otimes} \mathcal{C}^0(X_{\overline{K}_{S_0}} \times F_v^\times, E)\right)^{\mathrm{GL}}$ one gets that there would be a nonzero $v \in \left(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E) \hat{\otimes} \mathcal{C}^0(X_{\overline{K}_{S_0}} \times F_v^\times, \mathcal{O}_E)\right)^{\mathrm{GL}_2(F_v)}$ by rescaling. There must be an s such that $v \not\equiv 0 \pmod{\varpi_E^s}$, but reducing v mod ϖ_E^s, one gets that $v \equiv \sum c_i \otimes f_i \pmod{\varpi_E^s}$. But the f_i are smooth when reduced mod ϖ_E^s, so the c_is must be as well. But then one has that $v \equiv 0 \pmod{\varpi_E^s}$, a contradiction.

At this point, we have shown that the first and fourth terms in the exact sequence above are both 0, and so this gives an isomorphism between the second and third terms. To finish the analysis of the exact sequence, one needs to simplify the following two objects:

$$\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s \varinjlim_n H^1_{\acute{e}t}(\Gamma_i'\backslash\Sigma^n,\mathcal{O}_E/\varpi_E^s)\right)\otimes_{\mathcal{O}_E} E\right) \text{ and} \tag{2}$$

$$\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s \varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)^{\Gamma_i'}\right)\otimes_{\mathcal{O}_E} E\right). \tag{3}$$

Notice that, by definition of Γ_i', there is an isomorphism $Sh_{K_v^n K_{S_0} K_0^S,\mathbb{C}_p} \equiv \coprod_{\Gamma_i'} \Gamma_i'\backslash\Sigma^n$. Moreover, this decomposition respects connected components. Thus, one has that $\bigoplus_{\Gamma_i'} H^1_{\acute{e}t}(\Gamma_i'\backslash\Sigma^n,\mathcal{O}_E/\varpi_E^s) = H^1_{\acute{e}t}(Sh_{K_v^n K_{S_0} K_0^S,\mathbb{C}_p},\mathcal{O}_E/\varpi_E^s)$, and consequently, the term in (2) is $\hat{H}^1_{E,G}(K_{S_0})$.

Term (3) can be simplified as follows:

$$\begin{aligned}
&\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s \varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)^{\Gamma_i'}\right)\otimes_{\mathcal{O}_E} E\right)\\
=&\bigoplus_{\Gamma_i}\left(\left(\varprojlim_s \varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}^0(\Gamma_i\backslash \mathrm{GL}_2(F),\mathcal{O}_E/\varpi_E^s)^{\mathrm{GL}_2(F_v)}\right)\otimes_{\mathcal{O}_E} E\right)\\
=&\left(\left(\varprojlim_s \varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}^0(X_{\overline{K}_{S_0}},\mathcal{O}_E/\varpi_E^s)^{\mathrm{GL}_2(F_v)}\right)\otimes_{\mathcal{O}_E} E\right)\\
=&\left(\hat{H}^1_E(\Sigma)\hat{\otimes}_E \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})\right)^{\mathrm{GL}_2(F_v)}.
\end{aligned}$$

Putting these two equations together, one gets

$$\hat{H}^1_{E,G}(K_{S_0}) \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})\right)^{\mathrm{GL}_2(F_v)}. \tag{$*$}$$

As remarked, this is an isomorphism as $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[D^\times_{F_v}\times G_L]$-modules. We now prove the main theorems of the section.

Theorem 5.3.2. *Let $\mathfrak{p}\in \mathrm{maxSpec}(\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\frac{1}{p}])$ and $\pi_{\mathfrak{p},K_{S_0}} = \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}]$ (so that $\pi_{\mathfrak{p},K_{S_0}}$ is a $GL_2(F_v)$-representation). Then there is an isomorphism $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}] \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}\pi_{\mathfrak{p},K_{S_0}}\right)^{GL_2(F_v)}$.*

The proof of this is immediate from $(*)$ and the knowledge of what structures are respected under the isomorphism there. This does not immediately show that $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}]$ depends only on $\rho^m_S(\mathfrak{p})|_{G_F}$ and K_{S_0}, as it is not clear that $\pi_{\mathfrak{p},K_{S_0}}$ depends only on $\rho^m_S(\mathfrak{p})|_{G_{F_v}}$ and K_{S_0}. However, combining Theorem 5.3.2 with Theorem 4.3.1, one gets the following stronger theorem in the $F = \mathbb{Q}_p$ case.

Theorem 5.3.3. *If $F = \mathbb{Q}$ and thus $F_v = \mathbb{Q}_p$, there is an isomorphism*

$$\hat{H}^1_{E,G,\overline{\rho},S}[\mathfrak{p}] \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}B(\rho^m_S(\mathfrak{p}))\right)^{GL_2(\mathbb{Q}_p)}\overset{\curlywedge}{\otimes}\pi_{S_0}(\rho^m_S(\mathfrak{p})).$$

Finally, if one defines $J(\rho) := \mathrm{Hom}\left(\rho, \left(\hat{H}^1_E(\Sigma)\hat{\otimes}B(\rho)\right)^{\mathrm{GL}_2(\mathbb{Q}_p)}\right)$, then the following corollary is immediate from Theorem 5.3.3:

Corollary 5.3.4. *Let ρ be a 2-dimensional promodular E-representation of $G_{\mathbb{Q}}$. Assume further that $\overline{\rho}$ satisfies all of the running assumptions. Then there is an isomorphism*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G,\overline{\rho},S}) \cong J(\rho|_{G_{\mathbb{Q}_p}})\overset{\curlywedge}{\otimes}\pi_{S_0}(\rho).$$

6. Locally Algebraic Vectors

Throughout this section, the assumption that E is "large enough" will include the assumption that all algebraic representations of $D^\times_{F_v}$ as an algebraic group over $\mathbb{Q}_p$ that are defined over $\overline{\mathbb{Q}_p}$ are defined over E. This is equivalent to the assumption that there is a fixed embedding $E \to \overline{\mathbb{Q}_p}$ such that every embedding $\iota : F_v \to \overline{\mathbb{Q}_p}$ factors through $E \to \overline{\mathbb{Q}_p}$ and for every ι there is a field H_ι such that $\iota(F) \subset H_\iota \subset E$ and $[H_\iota : \iota(F)] = 2$.

The aim of this section is to describe the locally algebraic vectors in the representations $J(\pi)$ in the $F_v = \mathbb{Q}_p$ case and $J'(\pi)$ in the general case. In order to describe the main result, we will introduce a Jacquet-Langlands map from finite length locally algebraic representations of $\mathrm{GL}_2(F_v)$ to finite length locally algebraic representations of $D^\times_{F_v}$. Since $D^\times_{F_v} \times_{\mathbb{Q}_p} E = \prod_{\iota:F_v\to E}\mathrm{GL}_2/E$, the algebraic representations of $D^\times_{F_v}$

and $\mathrm{GL}_2(F_v)$ over E are naturally identified. Every finite length locally algebraic representation of $\mathrm{GL}_2(F_v)$ is of the form $\bigoplus_i \pi_i \otimes W_i$, where π_i is an indecomposable smooth representation of $\mathrm{GL}_2(F_v)$ and W_i is an algebraic representation of $\mathrm{GL}_2(F_v)$. There are 5 possabilities for π_i: π_i may be a character, an irreducible principle series, a twist of the Steinberg representation, an extension of a character by the Steinberg, or a supercuspidal representation.

Definition 6.0.5. *Let* $\pi = \bigoplus_i \pi_i \otimes W_i$ *as above. Define* $JL(\pi_i \otimes W_i)$ *to be* 0 *unless* π_i *is either a twist of the Steinberg representation or a spercuspidal representation. If* $\pi_i = (\chi \circ det) \otimes St$*, then define* $JL(\pi_i \otimes W_i) = (\chi \circ \nu) \otimes W_i$*, where* W_i *is viewed as a representation of* $D_{F_v}^\times$ *as above. If* π_i *is a supercuspidal representation, then define* $JL(\pi_i \otimes W_i) = JL^{cl}(\pi_i) \otimes W_i$*, where* JL^{cl} *is the classical Jacquet-Langlands correspondence for smooth supercuspidal representations of* $GL_2(F_v)$*. Finally, define* $JL\left(\bigoplus_i \pi_i \otimes W_i\right) = \bigoplus_i JL(\pi_i \otimes W_i)$.

The following is the main result of this section:

Theorem 6.0.6.

- *If* $F = \mathbb{Q}$*, assume that* $\rho : G_{\mathbb{Q}} \to GL_2(E)$ *is as in the introduction. Then* $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$.
- *For a general* L/F *as in section 3, one has that, if* $\rho : G_L \to GL_2(E)$ *arises in the cohomology of* $Sh_{K_v K_{S_0} K_0^S}$*, and* π *is the* ρ*-part of* $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$*, then* $J'(\pi)^{alg} = JL(\pi^{alg})$.

This section is similar to section 4.2 in [14], but there are three main remarks to be made. First of all, several of the results will be slightly less explicit. This is primarily an issue of focus: the aim here is to determine what the locally algebraic vectors are, whereas the aim in [14] is to elucidate the structure of the eigencurve. Secondly, there is no difference between compactly supported cohomology and regular cohomology in this case. This means that the arguments that were specific to H_c^i or H^i in [14] will now be needed to be made at the same time. Finally, while one might imagine that differences between the groups GL_2 and $D^\times$ would come up, this is not the case. The group theoretic properties of GL_2 that are used in [14] and thus the properties of $D^\times$ that are used in this paper are all about the algebraic representation theory. Since the algebraic representation theory of a group over a sufficiently large base field depends only on the geometric isomorphsim type of the group, none of those arguments will change.

6.1. **Notation and Elementary Calculations.** For this section, $\mathfrak{d}_{F_v}$ will be the Lie algebra of $D^\times_{F_v}$. As remarked before, $D^\times_{F_v}$ will be viewed as a group over $\mathbb{Q}_p$. Let $\mathfrak{sd}_{F_v}$ be the Lie algebra of the group $SD^\times_{F_v} := \{d \in D_{F_v} | \nu(d) = 1\}$ and $\mathfrak{z}_{F_v}$ be the Lie algebra of $Z(D^\times_{F_v}) = F_v^\times$. As remarked at the start of this section, one has that $\mathfrak{d}_{F_v}$ is a form of $\mathfrak{gl}_{2,F_v}$ and $\mathfrak{sd}_{F_v}$ is a form of $\mathfrak{sl}_{2,F_v}$. There is a natural isomorphism $\mathfrak{d}_{F_v} = \mathfrak{sd}_{F_v} \oplus \mathfrak{z}_{F_v}$. This isomorphism arises from the group homomorphism $SD^\times_{F_v} \times Z(D^\times_{F_v}) \to D^\times_{F_v}$ sending $(d, x) \to dx$, which has finite kernel and cokernel. Notice that $\mathfrak{z}_{F_v}$ is abelian, as it is the Lie algebra of an abelian group, and $\mathfrak{sd}_{F_v}$ is semisimple, as it is a form of a semisimple Lie group. Let $H^i(\mathfrak{g}; W) = Ext^i_{\mathfrak{g}}(\check{W}, E)$; this is the Lie algebra cohomology of $\mathfrak{g}$. An important result is the following:

Proposition 6.1.1. *Let W be an irreducible algebraic representation of $SD^\times_{F_v}$. Then one has that $H^i(\mathfrak{sd}_{F_v}; W) = 0$ unless W is the trivial representation, $3|i$, and $i \leq 3\deg(F/\mathbb{Q}_p)$.*

Proof. For irreducible representations W of $SD^\times_{F_v}$, there is a tensor product decomposition $W = \bigotimes_{\iota: F_v \hookrightarrow E} W_\iota$, where the terms W_ι are the base change of irreducible representations of $SD^\times_{F_v}$ as a group over F_v to E along ι. On the level of Lie algebras, this decomposition arises becuase one has that $\mathfrak{sd}_{F_v} \otimes_{\mathbb{Q}_p} E = \bigoplus_\iota \mathfrak{sd}_{F_v} \otimes_{\iota(F_v)} E$. As a notational convience, let $\mathfrak{sd}_\iota$ be the summands of the direct sum decomposition of $\mathfrak{sd}_{F_v} \otimes_{\mathbb{Q}_p} E$. There is a Künneth formula, showing that $H^*(\mathfrak{sd}_{F_v}; W) =$

$\bigoplus_\iota H^*(\mathfrak{so}_\iota; W_\iota)$. Thus, the statement in the theorem is reduced to the statement that, for any irreducible representation W_ι of $\mathfrak{so}_\iota$, then $H^i(\mathfrak{so}_\iota, W_\iota) = 0$ unless $i = 0$ or 3 and W_ι is trivial.

$H^0(\mathfrak{so}_\iota; W_\iota) = W_\iota^{\mathfrak{so}_\iota}$. This is visibly 0 unless W_ι is the trivial representation. Since $\mathfrak{so}_\iota$ is semisimple, there are no non-trivial extensions between finite dimensional representations of $\mathfrak{so}_\iota$, so $H^1(\mathfrak{so}_\iota; W_\iota) = 0$ for all W_ι. Poincaré duality says that $H^3(\mathfrak{so}_\iota; E) = E$, that the cup product pairing from $H^i(\mathfrak{so}_\iota; W_\iota) \times H^{3-i}(\mathfrak{so}_\iota; \check{W}_\iota) \to H^3(\mathfrak{so}_\iota; E)$ is a perfect pairing for $0 \leq i \leq 3$, and that $H^i(\mathfrak{so}_\iota; W_\iota) = 0$ for all $i > 3$. Since $H^1(\mathfrak{so}_\iota, \check{W}_\iota) = 0$ for all W_ι, one has that $H^2(\mathfrak{so}_\iota; W_\iota) = 0$. If W_ι is not the trivial representation, then $H^0(\mathfrak{so}_\iota; \check{W}_\iota) = 0$ so $H^3(\mathfrak{so}_\iota; W_\iota) = 0$ as well. Finally, if $i > 3$, then $H^i(\mathfrak{so}_\iota; W_\iota) = 0$ directly because of Poincaré duality. □

There are also a couple important results that are needed in order to apply the results in [14].

Proposition 6.1.2. *$\hat{H}^i_{\mathcal{O}_E,G}(K_{S_0})$ is the ϖ_E-adic completion of $\varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E)$.*

Proof. $Sh_{K_{S_0}K_p}$ is a complete curve, and thus one has that $H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E^s = H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)$. This remains true when passing to the limit along the K_ps, and since the ϖ_E-adic completion of $\varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E)$ is definitionally $\varprojlim_s \varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E^s$, the result follows. □

The importance of this proposition is that it identifies what we denote $\hat{H}^i$ with what is denoted $\hat{H}^i$ in [14]. The other result is the following calculation:

Lemma 6.1.3. *The space $\hat{H}^2_{E,G}(K_{S_0})$ vanishes.*

Proof. We will show that, for all K_p and s, there is a compact open subgroup $K'_p \subset K_p$ such that the map from $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p,\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s) \to H^2_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)$ is 0. This implies that $\varinjlim_{K_p} H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s) = 0$ and thus that $\hat{H}^2_{E,G}(K_{S_0}) = 0$.

Fixing K_p and s, choose $K'_p \subset K_p$ such that $p^s | [(K'_p \cap SD^\times) : (K_p \cap SD^\times)]$. $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p}, \mathcal{O}_E/\varpi_E^s)$ is freely generated by classes $[C']$ for each connected curve C' of $Sh_{K_{S_0}K_p}$, and $H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s)$ is freely generated by by classes $[C]$ for each connected curve C of $Sh_{K_{S_0}K_p}$. If $C' \to C$ with degree d, then the image of $[C']$ in $H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s)$ is $d[C]$. But, over a fixed connected curve C of $Sh_{K_{S_0}K_p}$, the map from $Sh_{K_{S_0}K'_p} \to Sh_{K_{S_0}K_p}$ sends $[K'_p : K_p]/[(K'_p \cap SD^\times) : (K_p \cap SD^\times)]$ connected curves C' onto C with degree $[(K'_p \cap SD^\times) : (K_p \cap SD^\times)]$. Thus, the image of $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p}, \mathcal{O}_E/\varpi_E^s)$ in $H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s)$ is divisible by p^s, and thus ϖ_E^s, and thus is 0. □

Lemma 6.1.4. *There is an integer r and a compact open subgroup $H \subset F_v^\times$ such that $\hat{H}^0_{E,G}(K_{S_0}) \cong \mathcal{C}^0(H, \mathcal{O}_E)^r$.*

Proof. In the $F = \mathbb{Q}$ case, the connected components of $Sh_{K_pK_{S_0}K_0^S}$ are parameterized by the double coset space $\mathbb{R}_+ \times \mathbb{Q}^\times \backslash \mathbb{A}^\times / \nu(K_pK_{S_0}K_0^S)$. It is well known that $\mathbb{A}^\times = \mathbb{R}_+ \times \mathbb{Q}^\times \times \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \times \mathbb{Z}_p^\times$. If we let $S^p = \prod_{\ell \neq p} \mathbb{Z}_\ell^\times / \nu(K_{S_0})$ and $r = \#(S^p)$, then one gets that the connected components are parameterized by $S^p \times (\mathbb{Z}_p^\times / \nu(K_p))$. Additionally, as K_p shrinks to $\{1\}$, $\nu(K_p)$ shrinks to $\{1\}$ as well. An argument entirely analogous to proposition 4.1.1 shows that $\hat{H}^0_{\mathcal{O}_E,G}(K_{S_0}) = \mathcal{C}^0(\mathbb{Z}_p^\times \times S^p, \mathcal{O}_E) = \mathcal{C}^0(\mathbb{Z}_p^\times, \mathcal{O}_E)^r$.

In the other case, let $T = \{x \in L^\times | x\overline{x} = 1\}$, a torus over F that is the determinant group of G. Then the connected components of $Sh_{K_vK_{S_0}}$ are parameterized by the double coset space $T(F) \backslash T(\mathbb{A}_{F,f}) / \det(K_vK_{S_0})$. Choose $H \subset F_v^\times = T(F_v)$ sufficiently small that $T(\mathcal{O}_F) \cap H\det(K_{S_0}) = \{1\}$. If $S^v = T(F) \backslash T(\mathbb{A}_{F,f}) / H\det(K_{S_0})$ and $r = \#(S^v)$, then the description of the connected components shows that, for K_v sufficiently small, the connected components are parameterized by $S^v \times H/\det(K_v)$ as an H-set. Now, a similar argument as before shows that $\hat{H}^0_{\mathcal{O}_E,G}(K_{S_0}) = \mathcal{C}^0(H, \mathcal{O}_E)^r$. □

Corollary 6.1.5. *Viewing $\hat{H}^0_{E,G}(K_{S_0})$ as a representation of $SD_{F_v}^\times \times Z(D_{F_v}^\times)$, one has that $\hat{H}^0_{E,G}(K_{S_0}) = 1_{SD_{F_v}^\times} \boxtimes \hat{H}^0_{E,G}(K_{S_0})|_{Z(D_{F_v}^\times)}$.*

Proof. This corollary is equivalent to the triviality of the action of $SD_{F_v}^{\times}$ on $\hat{H}^0_{E,G}(K_{S_0})$. However, this is exactly the content of 6.1.4. $\square$

Proposition 6.1.6. *Let W be an irreducible algebraic representation of $D^{\times}$. Then*

$$\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) = 0$$

unless W is one dimensional, $3|i$, and $i \leq 3\deg(F/\mathbb{Q})$

Proof. Let χ be the central character of W. One may then write $\check{W} = E(\chi^{-1}) \boxtimes \check{W}|_{\mathfrak{sd}_{F_v}}$ where the first factor is a representation of $\mathfrak{z}_{F_v}$. Since the action of $\mathfrak{sd}_{F_v}$ on $\hat{H}^0_{E,G}(K_{S_0})^{la}$ is trivial, one may write $\hat{H}^0_{E,G}(K_{S_0})^{la} = \hat{H}^0_{E,G}(K_{S_0})^{la}|_{\mathfrak{z}_{F_v}} \boxtimes E$ where the second factor is the trivial representation of $\mathfrak{sd}_{F_v}$. Now, one can appeal to the Künneth formula and get

$$\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) = \bigoplus_{a+b=i} \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E(\chi^{-1}), \hat{H}^0_{E,G}(K_{S_0})^{la}) \otimes \mathrm{Ext}^b_{\mathfrak{sd}_{F_v}}(\check{W}, E).$$

The second term is equal to $H^b(\mathfrak{sd}_{F_v}; W)$, which is zero unless $b = 0$ or 3 and $\dim(W) = 1$. Thus, we may assume that $\dim(W) = 1$. To compute the first term, choose a compact $H \subset Z(D^{\times})$ sufficiently small so that $\nu|_H : H \to F_v^{\times}$ is an isomorphism onto its image and such that $\hat{H}^0_{E,G}(K_{S_0}) = \mathcal{C}^0(H, E)^r$ as H-representations for some integer r. Then there is a sequence of isomorphisms

$$\begin{aligned}
\mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E(\chi^{-1}), \hat{H}^0_{E,G}(K_{S_0})^{la}) &\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, \hat{H}^0_{E,G}(K_{S_0})^{la} \otimes E(\chi)) \\
&\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, (\mathcal{C}^0(H, E)^{la})^r \otimes E(\chi)) \\
&\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, (\mathcal{C}^0(H, E)^{la})^r) \\
&\cong H^a(\mathfrak{z}_{F_v}; (\mathcal{C}^0(H, E)^{la})^r).
\end{aligned}$$

The third isomorphism arises because $f(h) \mapsto \chi(h)f(h)$ is an automorphism on $\mathcal{C}^0(H,E)$, and the fourth isomorphism is an alternative definition of Lie algebra cohomology. Then by theorems 1.1.12 (v) and 1.1.13 in [14], one sees that $H^a(\mathfrak{z}_{F_v}; (\mathcal{C}^0(H^E)^{la})^r) = 0$ unless $a = 0$. Thus, the only nonzero terms in the Künneth description are when $a = 0$ and $b = 0$ or $b = 3$. That is, $\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la})$ is only nonzero when $3|i$ and $I \leq 3\deg(F/\mathbb{Q})$. □

6.2. **Conculsions About Locally Algebraic Vectors.** The following notation will be useful in this section. Recall that there is a local system $\mathcal{V}_W$, defined in [10] for the $F = \mathbb{Q}$ case and in (much) more generality in e.g. [14], on $Sh_{K_{S_0}K_p}$. We will let $H^i(\mathcal{V}_W, K_{S_0}) = \varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{V}_W)$. This is a smooth E representation of $D^\times$ that can be understood in terms of automorphic forms. Because the Shimura curve is 1-dimensional, one has that $H^i(\mathcal{V}_W, K_{S_0})$ vanishes if $i > 2$. Additionally, Corollary 2.2.18 in [14] constructs a spectral sequence starting on the E_2-page with $\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^j_{E,G}(K_{S_0})^{la}) \Rightarrow H^{i+j}(\mathcal{V}_W, K_{S_0})$. Since the $E_2^{i,j}$ terms are zero for $j > 2$, the spectral sequence collapses on the E_3 page. Proposition 6.1.6 says that the $E^{i,0}$ terms are 0 unless $3|i$ and $i \leq 3\deg(F/\mathbb{Q})$. Since $H^n(\mathcal{V}_W, K_{S_0}) = 0$ for $n \geq 3$, one has that $E_2^{i,1} = 0$ for $i \geq 2$. Additionally, one has that $d_3 : E_3^{1,1} \to E_3^{3,0}$ is surjective with kernel H^2, that $E_3^{0,1} = H^1$, and that $E_3^{0,0} = H^0$. Unwinding the terms in the spectral sequence, one has the following equalities:

$$H^0(\mathcal{V}_W, K_{S_0}) = \mathrm{Hom}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}), \tag{4}$$

$$H^1(\mathcal{V}_W, K_{S_0}) = \mathrm{Hom}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}), \text{ and} \tag{5}$$

(6)

$$\mathrm{Ext}^{3j-2}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) = \mathrm{Ext}^{3j}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}). \qquad (2 \leq j \leq \deg(F/\mathbb{Q}))$$

and the following short exact sequence:

$$0 \to H^2(\mathcal{V}_W, K_{S_0}) \to \mathrm{Ext}^1_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) \to \mathrm{Ext}^3_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) \to 0.$$

Finally, notice that, unless W is one-dimensional, the left-hand term and hence the right-hand term in (4) is 0.

For the reader that is unfamiliar with spectral sequences, here is a picture of the E_2-page with (possibly) non-zero terms written as $E_2^{i,j}$ to keep in mind when reading the above:

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \dots \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & \dots \\
E_2^{0,0} & 0 & 0 & E_2^{3,0} & 0 & \dots
\end{array}$$

Recall the following fact proved using local-global compatability at p and multiplicity one results:

Fact 6.2.1. *Let W have weights $\{(w_{\iota,1}, w_{\iota,2})\}$ where $w_{\iota,1} < w_{\iota,2}$. Then one has*

$$H^1(\mathcal{V}_W, K_{S_0}) = \bigoplus_{\rho} \rho \otimes JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes \left(\bigotimes_{\ell \neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}},$$

with the direct sum running over all G-modular representations ρ of $G_{\mathbb{Q}}$ in the $F = \mathbb{Q}$ case and G_L case for general F with Hodge-Tate weights $-w_{\iota,2} - 1$ and $-w_{\iota,1}$.

The following precise formulations of Theorem 6.0.6 can now be proved:

Theorem 6.2.2. *Let $\rho : G_{\mathbb{Q}} \to GL_2(E)$ be a promodular representation of $G_{\mathbb{Q}}$ unramified outside of S. Assume further that $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ is an extension of a character χ by $\chi\epsilon$ for all $\ell \neq p$ such that G is ramified at ℓ. Then one has that $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$.*

Proof. Consider the space $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg})$. On one hand, this is equal to $\left(J(B(\rho|_{G_{\mathbb{Q}_p}})) \otimes \left(\bigotimes_{\ell\neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})\right)^{K_{S_0}}\right)^{alg}$ or equivalently, since being locally algebraic only depends on the $D^\times_{\mathbb{Q}}$-action, $\left(J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \otimes \left(\bigotimes_{\ell\neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})\right)^{K_{S_0}}\right)$. On the other hand, this is the union of the spaces $\check{W} \otimes \mathrm{Hom}_{G_{\mathbb{Q}},\mathfrak{d}_{F_v}}(\rho\otimes\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la})$ over all algebraic representations W. As above, one has that there is an equality $\mathrm{Hom}_{G_{\mathbb{Q}},\mathfrak{d}_{F_v}}(\rho\otimes\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, H^1(\mathcal{V}_W))$.

If ρ is G-modular, then $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$ and moreover one has that $WD(\rho|_{G_{\mathbb{Q}_p}})$ is indecomposable. In this case, one has that ρ arises in $H^1(\mathcal{V}_{W_0})$ where W_0 has weights $-w_2 \leq -w_1 - 1$ and only for this W_0. For this W_0, $\check{W}_0$ has weights w_2 and w_1+1, which means that $\check{W}_0 = \mathrm{Sym}^{w_2-w_1-1}(\mathrm{Std})\otimes\det^{w_1+1}$. Importantly, $B(\rho|_{G_{\mathbb{Q}_p}})^{alg} = \pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}}))\otimes\check{W}$. Thus, one has that $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg}) = JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}}))\otimes\check{W}$. Plugging this into $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg})$, the following chain of equalities holds:

$$
\begin{aligned}
&\left(J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \otimes \left(\bigotimes_{\ell\neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})\right)^{K_{S_0}}\right)\\
&= \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg})\\
&= \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, H^1(\mathcal{V}_W, K_{S_0}))\otimes\check{W}\\
&= JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}}))\otimes\left(\bigotimes_{\ell\neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})\right)^{K_{S_0}}\otimes\check{W}\\
&= JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})\otimes\left(\bigotimes_{\ell\neq p}{}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})\right)^{K_{S_0}}.
\end{aligned}
$$

Finally, it will suffice to show that if ρ is not G-modular, then $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg}$ and $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$ are both 0. Notice that, in the above argument, if ρ is not G-modular, then $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, H^1(\mathcal{V}_W)) = 0$ for all W, and so $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg}) = 0$ which implies that $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = 0$.

One has that $B(\rho|_{G_{\mathbb{Q}_p}})^{alg}$ is non-zero if and only if $B(\rho|_{G_{\mathbb{Q}_p}})$ is potentially semistable with disticnt Hodge-Tate weights. Local-global compatability for $\overline{G}$ implies that ρ is $\overline{G}$-modular. Moreover, if ρ is not G-modular, one gets that $WD(\rho|_{G_{\mathbb{Q}_p}})$ is the sum of two characters, and thus $\pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}}))$ is either a principal series or the extension of the Steinberg by a character. But then one has that $JL(\pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}}))\otimes W) = 0$ for all algebraic representations W of $\mathrm{GL}_2(\mathbb{Q}_p)$, which implies that $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg}) = 0$, showing the theorem. □

Theorem 6.2.3. *Let $\rho : G_L \to GL_2(E)$ be a G-promodular representation. If π is the ρ-part of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$, then $J'(\pi)^{alg} = \rho \otimes JL(\pi^{alg})$.*

The proof is almost identical to the above proof, with the weaker results coming from the weaker Theorem 5.3.2.

6.3. **Conclusions about the representations $J(\pi)$.** At this point, all results will be specialized to the $F = \mathbb{Q}$ case. The main result to be shown is that $(\hat{H}^1_{E,G,\overline{\rho},S})^{alg}$ is dense in $\hat{H}^1_{E,G,\overline{\rho},S}$. In fact, a stronger version of that will be shown:

Theorem 6.3.1. *Let K_{S_0} be sufficiently small. Then $\hat{H}^1_{E,G}(K_{S_0})^{\mathcal{O}_D^\times-alg}_{\overline{\rho}}$ is dense in $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}}$.*

Proof. Let K_{S_0} be small enough that the action of $G(\mathbb{Q})$ on $G(\mathbb{A}_f) \times \mathbb{H}^\pm / K_p K_{S_0} K_0^S$ is fixed point free for all compact open $K_p \subset D^\times$. Additionally, choose K_p small enough that K_p is normal in $\mathcal{O}_D^\times$ and K_p is pro-p. The first goal is to show that $\hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}}$ is injective as a smooth $(\mathcal{O}_E/\varpi_E^s)[K_p]$-module.

To that end, let M be a smooth finitely generated $(\mathcal{O}_E/\varpi_E^s)[K_p]$-module. Using the smallness assumption on K_{S_0}, there is a local system $\mathcal{M}$ over $Sh_{K_pK_{S_0}K_0^S}$ aassociated to M. One has that $H^i_{\acute{e}t}(Sh_{K_pK_{S_0}K_0^S,\overline{\mathbb{Q}}}, \mathcal{M})$ is a $\mathbb{T}(K_{S_0})$-module. We claim that

$H^0(Sh_{K_pK_{S_0}K_0^S},\mathcal{M})_{\overline{\rho}} = 0$ and $H^2(Sh_{K_pK_{S_0}K_0^S},\mathcal{M})_{\overline{\rho}} = 0$. By Poincaré duality, it is sufficient to show that $H^0(Sh_{K_pK_{S_0}K_0^S},\mathcal{M})_{\overline{\rho}} = 0$ for all M. Moreover, since $\overline{\rho}$ is irreducible, and the action of $\mathbb{T}(K_{S_0})$ on H^0 is through only reducible representations, one must have $H^0(Sh_{K_pK_{S_0}K_0^S},\mathcal{M})_{\overline{\rho}} = 0$. This implies that $M \mapsto H^1(Sh_{K_pK_{S_0}K_0^S},\mathcal{M})$ is an exact functor.

Now, we claim that $\mathrm{Hom}_{K_p}(M, \hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}}) = H^1(Sh_{K_pK_{S_0}K_0^S},\mathcal{M}^\vee)_{\overline{\rho}}$. Choose K_p' small enough that $M^{K_p'} = M$ and such that K_p' is normal in K_p. Then the Hochschild-Serre spectral sequence gives an exact sequence

$$\begin{aligned}
0 &\to H^1(K_p/K_p', H^0(Sh_{K_p'K_{S_0}K_0^S},\mathcal{M}^\vee)) \\
&\to H^1_{\acute{e}t}(Sh_{K_pK_{S_0}K_0^S\overline{\mathbb{Q}}},\mathcal{M}^\vee) \\
&\to H^1_{\acute{e}t}(Sh_{K_pK_{S_0}K_0^S\overline{\mathbb{Q}}},\mathcal{M}^\vee)^{K_p/K_p'} \\
&\to H^2(K_p/K_p', H^0(Sh_{K_p'K_{S_0}K_0^S},\mathcal{M}^\vee)).
\end{aligned}$$

Localizing at $\overline{\rho}$ kills the H^0-terms, so one gets an isomorphism

$$H^1(Sh_{K_pK_{S_0}K_0^S},\mathcal{M}^\vee)_{\overline{\rho}} \cong (H^1(Sh_{K_p'K_{S_0}K_0^S},\mathcal{M}^\vee)_{\overline{\rho}})^{K_p/K_p'}.$$

The assumption on K_p' implies that there is an equality as K_p/K_p'-representations:

$$H^1(Sh_{K_p'K_{S_0}K_0^S},\mathcal{M}^\vee) = \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, H^1(Sh_{K_p'K_{S_0}K_0^S},\mathcal{O}_E/\varpi_E^s)).$$

Plugging that into the above isomorphism, one gets an isomorphism

$$H^1(Sh_{K_pK_{S_0}K_0^S},\mathcal{M}^\vee)_{\overline{\rho}} \cong \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, H^1(Sh_{K_p'K_{S_0}K_0^S},\mathcal{O}_E/\varpi_E^s)_{\overline{\rho}})^{K_p/K_p'}.$$

Finally, as K_p' shrinks to $\{1\}$, one gets the isomorphism claimed at the start of the paragraph.

Thus, $M \mapsto \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[K_p]}(M, \hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}})$ is exact, as it is the composition of $M \mapsto \mathcal{M}^\vee \mapsto H^1(Sh_{K_pK_{S_0}K_0^S},\mathcal{M}^\vee)$, both of which are exact. Consequently,

$\hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}}$ is an injective $\mathcal{O}_E/\varpi_E^s$-module, so $(\hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}})^\vee$ is projective as a $\mathcal{O}_E[[K_p]]$-module. Since K_p is assumed to be pro-p, $(\mathcal{O}_E/\varpi_E^s)[[K_p]]$ is a local ring, so there is an integer r_s such that $(\hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}})^\vee \cong ((\mathcal{O}_E/\varpi_E^s)[[K_p]])^{r_s}$. Dualizing, that says that $\hat{H}^1_{\mathcal{O}_E/\varpi_E^s,G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, \mathcal{O}_E/\varpi_E^s)^{r_s}$.

Tensoring both sides with k_E, one gets that $\hat{H}^1_{k_E,G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, k_E)^{r_s}$. Thus, r_s doesn't depend on s and, after taking the inverse limit over s, there is an isomorphism $\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, \mathcal{O}_E)^r$. Inverting p, one gets $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, E)^r$. The map from $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\mathcal{O}_D^\times]])}((\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee, *)$ to $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[K_p]])}((\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee, *)^{\mathcal{O}_D^\times}$ is an isomorphism. Moreover, the second functor is exact as taking the invariants of a finite group is exact in characteristic 0. Thus, $(\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee$ is projective as an $E \otimes (\mathcal{O}_E[[\mathcal{O}_D^\times]])$-module. Thus, it is a summand of a free module, and thus, after dualizing, one has that $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}}$ is a summand of $\mathcal{C}^0(\mathcal{O}_D^\times, E)^t$ for some t. It is then sufficient to show that the $\mathcal{O}_D^\times$-algebraic vectors are dense in the $\mathcal{C}^0(\mathcal{O}_D^\times, E)$. But the $\mathcal{O}_D^\times$ algebraic vectors in $\mathcal{C}^0(\mathcal{O}_D^\times, E)$ are exactly the polynomial functions, and Mahler expansions express polynomials as a dense subspace of continuous functions. □

The above proof, in addition to being similar to the one in [15], models the same philosophy: H^1 becomes exact when localizing at a non-Eisenstein prime. In addition, the above proof shows that $\hat{H}^1_{E,G,\overline{\rho},S}$ is cofinitely generated as a $D^\times \times G_{S_0}$-representation.

Appendix A. Local Langlands for $\ell \neq p$

This is an expository section on the local Langlands correspondence in families, due to Emerton and Helm in [16]. The main goal is to study what happens in the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for smooth representations over p-adically complete rings. The motivating question is the following. Consider ρ_S^m as a representation of $G_\mathbb{Q}$ over $\mathbb{T}_{\overline{G},\overline{\rho},S}$. For $\mathfrak{p} \subset \mathbb{T}_{\overline{G},\overline{\rho},S}$ a characteristic 0 prime, there is a representation $\rho(\mathfrak{p}) : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(k(\mathfrak{p}))$ by restriction. This gives rise to a smooth representation $\pi(\mathfrak{p})$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. Is it possible to find a smooth representation π of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over $\mathbb{T}_{\overline{G},\bar{\rho},S}$ such that π specialized to $\mathfrak{p}$ is $\pi(\mathfrak{p})$? The results in this section will be stated in larger generality, but this example should be kept in mind throughout.

A.1. **The Kirillov Functor.** Recall that $G_{S_0} = \prod_{\ell \in S_0} G(\mathbb{Q}_\ell) = \prod_{\ell \in S_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$. The goal of this section is to study smooth representations of G_{S_0} over a ring A that is a complete noetherian local $\mathcal{O}_E$-algebra. Let

$$P_0 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \prod_{\ell \in S_0} \mathbb{Z}_\ell^\times, b \in \prod_{\ell \in S_0} \mathbb{Z}_\ell \right\} \subset G_{S_0}.$$

Also let $U_\ell \in \mathcal{O}_E[G_{S_0}]$ be $\displaystyle\sum_{i=0}^{\ell-1} \begin{pmatrix} \ell & i \\ 0 & 1 \end{pmatrix}$.

Definition A.1.1 (The Kirillov Functor)**.** *Let X be a smooth representation of G_{S_0} over A. The functor $F_{S_0}(X) := \{x \in X^{P_0} | U_\ell x = 0$ for all $\ell \in S_0\}$ sending smooth represntations of G_{S_0} over A to A-modules is called the Kirillov functor.*

The key properties of F_{S_0} are as follows (this may be found in [16])

Theorem A.1.2. *(1) F_{S_0} is exact.*

(2) F_{S_0} sends finite length smooth representations to finite length A-modules.

(3) If X/k is a smooth representation of G_{S_0}, then $F_{S_0}(X)$ is at most one dimensional.

(4) If S_0 is the singleton set $\{\ell\}$ and X/k is an irreducible representation of G_{S_0}, then $F_{S_0}(X) = 0$ if and only if X is a character of the determinant.

Definition A.1.3. *A representation X/k is generic if and only if there are no subrepresentations W of X with $F_{S_0}(W) = 0$.*

Lemma A.1.4. *Let $f : W \to X$ be a G_{S_0} equivariant map from one smooth admissible G_{S_0} representation over k to another. If W is generic and $F_{S_0}(f)$ is injective, then so is f.*

Proof. Let $V \subset W$ be the kernel of f and assume that $V \neq 0$. By genericity, one sees that $F_{S_0}(V) \neq 0$. Exactness of F_{S_0} implies that $F_{S_0}(f)(V) = 0$, contradicting the assumption that $F_{S_0}(f)$ is injective. □

A.2. **Generic Local Langlands for $\mathrm{GL}_2(\mathbb{Q}_\ell)$.** Let A be a complete noetherian $\mathcal{O}_E$-algebra. Assume further that A is a domain of characteristic 0, and let $\mathcal{K}$ be the field of fractions of A. In [16], they define a correspondence between continuous two-dimensional representations of $G_{\mathbb{Q}_\ell}$ over $\mathcal{K}$ and generic smooth representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over $\mathcal{K}$. This is usually given by the standard recipe: if ρ is a two-dimensional representation of $G_{\mathbb{Q}_\ell}$ over $\mathcal{K}$, then one passes to the associated Weil-Deligne representation $WD(\rho)$ and then to the Frobenius-semisimplification $WD(\rho)^{F-ss}$. From here, one appeals to the standard local Langlands correspondence and gets a representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. However, that representation is not always generic (recall that, in the case of $\mathrm{GL}_2(\mathbb{Q}_\ell)$, this means that the representation has no finite-dimensional subrepresentations). Thus, a small modification is made to make sure that one always gets a generic representation. In order to fix that, the following construction is made:

Definition A.2.1. *Let ρ be a two-dimensional continuous representation of $G_{\mathbb{Q}_\ell}$ as above. One defines a smooth admissible generic representation $\pi_{LL}(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over $\mathcal{K}$ as follows*

(1) *If ρ is absolutely irreducible, then $\pi_{LL}(\rho)$ is constructed as above. In particular, $\pi_{LL}(\rho)$ is a supercuspidal representation with central character given by $\det(\rho)|\cdot|_\ell$.*

(2) *If ρ is the sum of characters $\chi_1 \oplus \chi_2$ with $\chi_1\chi_2 \neq |\cdot|_\ell^{\pm 1}$, then $\pi_{LL}(\rho)$ is again constructed as above. In this case, $\pi_{LL}(\rho)$ is a principal series, given by taking $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi_1|\cdot|_\ell \boxtimes \chi_2)$. This representation is generic, irreducible, and insensitive to the choice of ordering of the characters.*

(3) *If ρ is the sum of characters $\chi_1 \oplus \chi_2$ with $\chi_1\chi_2 = |\cdot|_\ell^{\pm 1}$, choose χ_1 so that $\chi_1 = \chi_2|\cdot|_\ell$. Then $\pi_{LL}(\rho)$ is again a principal series given by $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi_1|\cdot|_\ell \boxtimes \chi_2)$. While this is the same formula as above, this representation is no longer irreducible, but it is instead a nonsplit extension of $\chi_1 \circ \det$ by $\chi_1 \circ \det \otimes St$. This will be the only case that differs from the standard construction.*

(4) *If ρ is a non-zero extension of χ by χ, then $\pi_{LL}(\rho)$ is $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_\ell)}(\chi|\cdot|_\ell \boxtimes \chi)$. Again, this is the same formula as number 2. In this case, this is due to the associated Weil-Deligne representation not being Frobenius semi-simple. The Frobenius semplification of $WD(\rho)$ is just the Weil-Deligne representation associated to $\chi \oplus \chi$.*

(5) *If ρ is a non-zero extension of $\chi|\cdot|_\ell$ by χ, then $\pi_{LL}(\rho)$ is just defined to be $(\chi|\cdot|_\ell) \circ \det \otimes St$.*

A.3. **Mod p Local Langlands for $\mathrm{GL}_2(\mathbb{Q}_\ell)$.** Before constructing the local Langlands correspondence in families, it is usefule to figure out what happens mod p. In [16], there is a construction of a correspondence $\overline{\rho} \to \overline{\pi}(\overline{\rho})$ sending two-dimensional representations of $G_{\mathbb{Q}_\ell}$ over k_E to smooth admissible representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$.

Theorem A.3.1. *The correspondence $\overline{\rho} \to \overline{\pi}(\overline{\rho})$ satisfies and is characterized by the following properties:*

(1) *$\overline{\pi}(\overline{\rho})$ is generic (again, this means that there are no finite-dimensional subrepresentations).*

(2) *For any K/E finite together with $\rho : G_{\mathbb{Q}_\ell} \to GL_2(\mathcal{O}_K)$ such that $\rho \otimes_{\mathcal{O}_K} k_K \cong \overline{\rho} \otimes_{k_E} k_K$, there is a lattice $\pi_{LL}(\rho)^\circ$ in $\pi_{LL}(\rho \otimes_{\mathcal{O}_K} K)$ such that $\pi_{LL}(\rho)^\circ \otimes_{\mathcal{O}_K}$*

$k_K \hookrightarrow \overline{\pi}(\overline{\rho}) \otimes_{k_E} k_K$. *Additionally such a lattice is unique up to choice of multiplication by scalar in* $K^\times$.

(3) *For any representation* $\overline{\pi}$ *that satisfies the above two conditions, there is a* $\mathrm{GL}_2(\mathbb{Q}_\ell)$*-equivariant embedding of* $\overline{\pi}(\overline{\rho}) \hookrightarrow \overline{\pi}$.

A.4. **Local Langlands in Families.** Recall that A is a complete noetherian local $\mathcal{O}_E$-algebra with maximal ideal $\mathfrak{m}$. For this section, assume further that A is reduced and flat over $\mathcal{O}_E$. For each prime ideal $\mathfrak{p} \subset A$, let $\kappa(\mathfrak{p})$ be the residue field of $\mathfrak{p}$. If ρ is a two-dimensional representation of $G_{\mathbb{Q}}$ over A, then $\rho(\mathfrak{p})$ will denote $\rho \otimes_A \kappa(\mathfrak{p})$. Additionall, $\overline{\rho}$ will be used to denote $\rho(\mathfrak{m})$. The main result is the following:

Theorem A.4.1. *Let* ρ *be as above. There is at most one representation* X *of* G_{S_0} *satisfying the following properties:*

(1) $X/\varpi_E X[\mathfrak{m}]$ *is generic.*

(2) $F_{S_0}(X/\varpi_E X[\mathfrak{m}])$ *is at most one-dimensional.*

(3) *There is a Zariski-dense set of closed primes* $\mathcal{S} \subset \mathrm{Spec}(A[\frac{1}{p}])$ *such that*

$$(X \otimes_{\mathcal{O}_E} E)[\mathfrak{p}] \cong \bigotimes_{\ell \in S_0} \pi_{LL}(\rho(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})$$

where π_{LL} *is the correspondence described in defintion A.2.1, and such that the closure in* X *of the saturation in* X *of* $\sum_{\mathfrak{p} \in \mathcal{S}} X[\mathfrak{p}]$ *is just* X.

Moreover, for such an X *and any closed prime* $\mathfrak{p} \in \mathrm{Spec}(A[\frac{1}{p}])$, *one has that there is a* G_{S_0}*-equivariant embedding*

$$(X \otimes_{\mathcal{O}_E} E)[\mathfrak{p}] \hookrightarrow \bigotimes_{\ell \in S_0} \pi_{LL}(\rho(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}).$$

If X satisfies the conditions of Theorem A.4.1 then we will say that $X = \pi_{S_0}(\rho)$.

Appendix B. The p-adic Langlands Correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

This is a short introduction to the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$. A good source to see the results (without proofs) is section 2 of [3]. Throughout this section, the following assumption will be in force:

Assumption B.0.2. *For any representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k_E)$, one has that*

$$\overline{\rho} \ncong \begin{pmatrix} 1 & * \\ & \overline{\epsilon} \end{pmatrix}.$$

B.1. **The mod p Theory.** The mod p local Langlands correspondence is a correspondence between smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ valued in k_E-vector spaces and two-dimensional representations of $G_{\mathbb{Q}_p}$ over k_E. The explicit constructions given here may seem unmotivated, but there is a characterization due to Colmez in [9] using (φ, Γ)-modules.

First, let χ_1 and χ_2 denote two k_E-valued characters of $G_{\mathbb{Q}_p}$.

Theorem B.1.1. *If $\chi_1\chi_2^{-1} \neq \overline{\epsilon}^{\pm 1}$ and $\chi_1 \neq \chi_2$, then one has that*

$$\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\chi_1, \chi_2) \cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2\overline{\epsilon}), \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1\overline{\epsilon})).$$

The assumptions on χ_1 and χ_2 might seem restrictive, but they are consistent with global assumptions made on the representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k_E)$. Additionally, they guarantee both that the principal series are irreducible and that the Ext-groups are one-dimensional. This theorem has a more general version that removes the assumptions on χ_1 and χ_2, but care needs to be made in those cases. In particular, one has that for $\pi = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi \boxtimes \chi)$ is not irreducible but rather is an extension of the form $0 \to (\chi \circ \det) \otimes St \to \pi \to \chi \circ \det \to 0$. Additionally, the dimension of the Ext-groups can jump to 2 in these cases.

In order to handle the irreducible case, let $\omega : I_{\mathbb{Q}_p} \to \mathbb{F}_{p^2}^\times$ be the fundemental character of niveau two. Concretly, $\omega(g) \equiv \frac{g\cdot(-p)^{\frac{1}{p^2-1}}}{(-p)^{\frac{1}{p^2-1}}} \pmod p$, which doesn't depend on choice of (p^2-1)-th root of $-p$. Any irreducible representation $\overline{\rho}$ of $G_{\mathbb{Q}_p}$ over k_E has inertial type $\overline{\rho}|_{I_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} \omega^{a+pb} & \\ & \omega^{pa+b}\end{smallmatrix}\right)$ for some a, b with $0 \leq a < b \leq p$. Let $\overline{\rho}_{a,b}$ be the unique representation of $G_{\mathbb{Q}_p}$ whose restriction to inertia is $\left(\begin{smallmatrix} \omega^{a+pb} & \\ & \omega^{pa+b}\end{smallmatrix}\right)$ and whose determinant is $\overline{\epsilon}^{a+b}$. Every irreducible representation is a twist of one of the $\overline{\rho}_{a,b}$s by an unramified character.

Let σ be an irreducible representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ over k_E. Then, a theorem of Barthel and Livné shows that $\mathrm{End}(\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\sigma)) = k_E[T]$ for some Hecke operator T. Letting $\pi(\sigma, 0) = \mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\sigma)/T$, one gets a smooth irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The beauty of this situation is that it is possilbe to classify all such σs in a way that corresponds to the classification of irreducible $\overline{\rho}$s as above. For any such σ, one has that $\sigma|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ factors through $\mathrm{GL}_2(\mathbb{F}_p)$ and is isomorphic to $\mathrm{Sym}^s(\mathrm{Std}) \otimes \det^t$ for $0 \leq s \leq p-1$ and $0 \leq t \leq p-2$. The only information left to determine σ is an unramified character of $\mathbb{Q}_p^\times$. One then gets a bijection by letting $s = b - a - 1$ and $t \equiv b + a \pmod{p-1}$, and matching up unramified characters.

Theorem B.1.2 (The Mod p Langlands Correspondence)**.** *There is a correspondence between smooth admissible representations $\overline{\pi}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E and two dimensional representaions $\overline{\rho}$ of $G_{\mathbb{Q}_p}$ over k_E defined as follows:*

(1) If $\overline{\rho}$ is a sum (resp. extension) of χ_1 and (resp. by) χ_2, then $\overline{\pi}$ is a sum (resp. extension) of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2\overline{\epsilon})$ and (resp. by) $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1\overline{\epsilon})$.

(2) If $\overline{\rho}$ is irreducible, then one uses the recipe in the previous paragraph. That is to say, for $\overline{\rho}_{a,b}$, one chooses $\sigma|_{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathrm{Sym}^{b-a-1}(\mathrm{Std}) \otimes \det^{b+a}$ and $\sigma(p) = I$, and extends this correspondence to arbitrary representations by twisting by the corresponding characters under class field theory.

B.2. **The Characteristic Zero Theory.** The main objects of study in the p-adic theory are a little more complicated than the main objects of study in the classical local Langlands program. The representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ will be E-Banach spaces.

Definition B.2.1. *A Banach space over E is a topological vector space B complete with respect to a norm $||\cdot||_B$ satisfying the usual properties. Giving a norm is equivalent to giving a unit ball $B^\circ \subset B$ that is a p-adically seperated and complete $\mathcal{O}_E$-module. A representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a Banach space B is said to be unitary if it preserves a norm giving the topology. It is said to be continuous if the evaluation map $\mathrm{GL}_2(\mathbb{Q}_p) \times B \to B$ is continuous. If B is a unitary continuous representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on B descends to a smooth action on $B^\circ/\varpi_E B^\circ$. Under the same assumptions on B, it is said to be admissible if $B^\circ/\varpi_E B^\circ$ is admissible in the sense of smooth representations.*

A word of warning: there are representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that are unitary but not continuous and representations that are continuous but not unitary. If B is a unitary continuous admissible Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $v \in B$, define $f_v : \mathrm{GL}_2(\mathbb{Q}_p) \to B$ by $f_v(g) = g \cdot v$. Then there are two important subspaces of B:

Definition B.2.2. *Let B be as above. Then make the following definitions:*

- *The locally analytic vectors in B are denoted B^{la}. This is defined to be the subspace $\{v \in B | f_v$ is an analytic function in a neighborhood of $I\}$.*
- *The locally algebraic vectors in B are denoted B^{alg}. This is defined to be the subspace $\{v \in B | f_v$ is an algebraic function in a neighborhood of $I\}$.*

The Lie algebra $\mathfrak{gl}_2$ acts on B^{la} as the deriative of the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. A theorem of Schneider and Teitelbaum in [21] shows that B^{la} is always dense in B if B is a continuous unitary admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. On the other hand, B^{alg} is usually 0.

Work of many people produced the following theorem:

Theorem B.2.3 (The p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$)**.** *There is an association $\rho \to B(\rho)$ sending 2-dimensional representations of $G_{\mathbb{Q}_p}$ to continuous unitary admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E and inverse $\pi \to \mathbb{V}(\pi)$ going in the other direction satisfying the following properties:*

(1) $\rho \cong \rho'$ if and only if $B(\rho) \cong B(\rho')$ if and only if $B(\rho)^{la} \cong B(\rho')^{la}$.

(2) $B(\rho)^\circ/\varpi_E B(\rho)^\circ$ corresponds to $\overline{\rho}$ up to semiplification under the mod p Langlands correspondence.

(3) $B(\rho)^{alg} \neq 0$ if and only if ρ is potentially semistable with distinct Hodge-Tate weights.

(4) Conversely, if ρ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$, then $B(\rho)^{alg} \cong \pi_{LL}(WD(\rho)^{F-ss}) \otimes \mathrm{Sym}^{w_2-w_1-1}(\mathrm{Std}) \otimes \det^{w_1}$.

(5) $B(\rho)$ is "as irreducible as" ρ. That is to say, if ρ is irreducible, then so is $B(\rho)$. If ρ is reducible but indecomposable, then so is $B(\rho)$. If ρ is semisimple then so is $B(\rho)$.

The conceptual framwork behind why the conjecture was initially made (especially the stuff about the locally algebraic vectors) is as follows: if ρ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$, then the information needed to recover ρ is the Weil-Deligne representation $WD(\rho)$, the Hodge-Tate weights $w_1 < w_2$, and the weakly admissible filtration on the Weil-Deligne representation. The locally algebraic representation $\pi_{LL}(WD(\rho)^{F-s.s.}) \otimes \mathrm{Sym}^{w_2-w_1-1}(\mathrm{Std}) \otimes \det^{w_1}$ recovers $WD(\rho)$ and $w_1 < w_2$. The belief that was borne out was that the weakly admissible filtration should correspond to a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant admissible norm. In particular, if ρ is crystabelline and absolutely irreducible, then the weakly admissible filtration is unique up to isomorphism and so there should be exactly one norm on $\pi_{LL}(WD(\rho)^{F-s.s.}) \otimes \mathrm{Sym}^{w_2-w_1-1}(\mathrm{Std}) \otimes \det^{w_1}$. The following theorem due to Berger and Breuil in [1] shows exactly that:

Theorem B.2.4 (Berger-Breuil). *Let ρ be crystabelline and absolutely irreducible with distinct Hodge-Tate weights $w_1 < w_2$. Then $B(\rho)$ is the universal unitary completion of $\pi_{LL}(WD(\rho)^{F-s.s.}) \otimes \mathrm{Sym}^{w_2-w_1-1}(\mathrm{Std}) \otimes \det^{w_1}$.*

B.3. **Deformation Theory.** The final facet of the p-adic Langlands program that needs to be mentioned is the deformation theory aspect. Let $\mathrm{Comp}(\mathcal{O}_E)$ be the category of complete noetherian local $\mathcal{O}_E$-algebras whose residue field is a finite extension of k_E. Then there are a number of deformation functors that need to be discussed.

Definition B.3.1. *Let $\overline{\rho}$ be a 2-dimensional representation of $G_{\mathbb{Q}_p}$ over k_E. Then define $\mathrm{Def}(\overline{\rho})$ to be the following category fibered in groupoids over $\mathrm{Comp}(\mathcal{O}_E)$: let $A \in \mathrm{Comp}(\mathcal{O}_E)$. Then $\mathrm{Def}(\overline{\rho})(A)$ has objects that are free rank 2 A-modules V with a continuous action of $G_{\mathbb{Q}_p}$, together with an isomorphism of $G_{\mathbb{Q}_p}$-representations $i : V \otimes_A A/\mathfrak{m} \to \overline{\rho} \otimes_{k_E} A/\mathfrak{m}$. The morphisms are the isomorphisms of A-modules that commute with the $G_{\mathbb{Q}_p}$-action and with the isomorphisms i.*

Definition B.3.2. *Let $\overline{\rho}$ be as above, and let $\overline{\pi}$ be the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that is attached to $\overline{\rho}$. Then define $\mathrm{Def}(\overline{\pi})$ to be the following category fibered in groupoids over $\mathrm{Comp}(\mathcal{O}_E)$: let A be in $\mathrm{Comp}(\mathcal{O}_E)$. Then $\mathrm{Def}(\overline{\pi})(A)$ has objects that are A-modules π that are also orthonormalizable admissible continuous representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, together with an isomorphism $i : \pi \otimes_A A/\mathfrak{m} \to \overline{\pi} \otimes_{k_E} A/\mathfrak{m}$. The morphisms are again isomorphisms of $\mathrm{GL}_2(\mathbb{Q}_p)$ representations that commute with the isomorphisms i.*

The p-adic Langlands correspondence induces a natural morphism $\mathrm{Def}(\overline{\pi}) \to \mathrm{Def}(\overline{\rho})$. While this morphism is not necessarily an isomorphism, there are natural subfunctors that make this an isomorphsim when restricted to them.

Definition B.3.3. *Let $\overline{\rho}$ and $\overline{\pi}$ be as before.*

(1) Let $\mathrm{Def}^{crys}(\overline{\rho})$ be the Zariski closure of the crystalline points in the generic fiber of $\mathrm{Def}(\overline{\rho})$.

(2) Let $\mathrm{Def}^*(\overline{\pi})$ *be the subfunctor where one has that, for* $\pi \in \mathrm{Def}(\overline{\pi})(A)$ *and* $\mathbb{V}(\pi) \in \mathrm{Def}(\overline{\rho})(A)$*, the central character* χ *of* π *corresponds to* $\det(\mathbb{V}(\pi))\epsilon$ *under class field theory.*

(3) Let $\mathrm{Def}^{crys}(\overline{\pi}) = \mathrm{Def}^*(\overline{\pi}) \times_{\mathrm{Def}(\overline{\rho})} \mathrm{Def}^{crys}(\overline{\rho})$.

The main result due to Kisin in [18] is the following

Theorem B.3.4. *With* $\overline{\rho}$ *and* $\overline{\pi}$ *as above, one has the following:*

- *The morphism* $\mathrm{Def}^*(\overline{\pi}) \to \mathrm{Def}(\overline{\rho})$ *is a fully faithful embedding.*
- *When restricted to* $\mathrm{Def}^{crys}(\overline{\pi}) \to \mathrm{Def}^{crys}(\overline{\rho})$*, it becomes an isomorphism.*

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