TANGENT SPACES OF CERTAIN SPECTRA

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ABSTRACT. We outline a construction in derived algebraic geometry which produces elements of the $K(n)$-local Picard group. As an example, we produce a new model of a spectrum called the determinantal sphere, and as time permits we discuss newly indicated patterns in $K(n)$-local homotopy theory.

1. $K$-LOCAL INVERTIBLE SPECTRA

Throughout, fix a prime $p$ (often it is useful for $p$ to be odd, or even “large”) and a finite positive integer height $n$. We in chromatic homotopy theory are very interested in a certain sequence of extraordinary complex-oriented homology theories called Morava $K$-theories. There is one such $K$-theory for each choice of $n$ and $p$, and it is valued in graded vector spaces over the graded field $K_\ast(pt) = \mathbb{F}_p[v_n^{\pm}]$. One of our primary goals is to understand what these functors are telling us about Spectra. The theory of localization shows us that these Morava $K$-theories aren’t really telling us about Spectra per se, but rather about the full subcategory of $K$-local spectra, written $\text{Spectra}_{K}$. This subcategory comes with a reflector $L_K$, which has the properties that $K_\ast(X) = K_\ast(L_KX)$ and that $K_\ast$ reflects isomorphisms in $\text{Spectra}_{K}$.

Each of these three categories has a symmetric monoidal product: Spectra has a smash product $\wedge$, VectorSpaces$_K$ has a tensor product $\otimes$, and $\text{Spectra}_{K}$ has a localized smash product given by the formula $X \hat{\wedge} Y = L_K(X \wedge Y)$. Because $K_\ast$ is a graded field, the homology theory $K_\ast$ has Künneth isomorphisms, meaning that it is a monoidal functor, and by definition of $\hat{\wedge}$, $L_K$ is also monoidal. So, in our quest to understand $K_\ast$ and $\text{Spectra}_{K}$, it is profitable to consider what information we can glean from these products.

One invariant of these categories determined by this data is their subcategories of invertible objects; after replacing objects with their isomorphism classes, these are called the “Picard groups” of these categories. Using the monoidalness of our various functors, we record what we have so far in a diagram:

$$
\begin{array}{ccc}
\text{Spectra} & \xrightarrow{L_K} & \text{Spectra}_K \\
\wedge & & K_\ast \\
\text{Pic} & \xleftarrow{P_{\text{ic}_n}} & \text{Pic}_n \\
\end{array}
$$

The Picard subcategory of VectorSpaces$_K$ is easy to identify: the tensor product satisfies $\dim(V \otimes W) = \dim V \cdot \dim W$, and so the invertible vector spaces are those which are 1-dimensional, or which are “lines”. The Picard group Pic of Spectra is easy enough to describe: it is $\mathbb{Z}$, generated by $S^1$ the 1-sphere. The Picard group Pic$_n$ of Spectra$_K$ is extremely complex (and hence interesting); essentially the only quantitative thing we know about it in general is that it is an extension of a finite group by a profinite $p$-group. There are explicit computations for $n = 1$ and for $n = 2$ with $p \geq 3$, but otherwise it remains very mysterious. However, we do at least have one powerful tool at our disposal to identify when a spectrum is $K$-locally invertible: a theorem of Hopkins, Mahowald, and Sadofsky asserts that the right-hand square above is a pullback. Since the vertical arrows are inclusions, we may translate this theorem as asserting that a spectrum $X$ is $K$-locally invertible exactly when $K_\ast(X)$ is a line.

If Pic is isomorphic to $\mathbb{Z}$, and $\mathbb{Z}$ isn’t a profinite $p$-group, then obviously there are new elements in Pic$_n$. What do they look like? Here’s one cute and easy new family — start by considering the following horizontal system of cofiber sequences:

$$1^{\text{Making this identification is harder that it sounds; the proof I’ve read requires knowing the degree-wise finite generation of the stable homotopy groups of spheres.}}$$
The colimit, pictured on the far right, is also a cofiber sequence. The topmost object is the colimit of a sequence of identity morphisms, so is simply $S^{-1}$. The middle object is the colimit along iterates of the map $p$, so is, by definition, the spectrum $S^{-1}[p^{-1}]$ with the $p$-self-map inverted. Lastly, the spectrum on the bottom does not have a familiar name, so we call it $M^0(p^\infty)$.

Working $K$-locally, however, we can see that the middle spectrum is contractible: the map $p$ on $S^{-1}[p^{-1}]$ is exactly multiplication by $p$ in $K_*$-homotopy, but since the coefficient ring $K_*$ is of characteristic $p$, this is the zero map. On the other hand, $p$ is required to be invertible, which can only mean $K_*S^{-1}[p^{-1}] = 0$. In turn, this means that the going-around map $M^0(p^\infty) \to S^0$ is a $K$-local equivalence, and so $M^0(p^\infty)$ is an invertible spectrum, albeit not a very interesting one.\(^2\)

A different way of at least detecting that $M^0(p^\infty)$ is $K$-locally invertible is to apply $K$-homotopy to the bottom row: each object in the sequence becomes a 2-dimensional $K_*$-vector space, and each map from one to the next is $- \cdot p = - \cdot 0$ on the $(-1)$-graded piece and the identity on the $0$-graded piece — this is exactly what the diagram of cofiber sequences is recording. Hence, the homology of the colimit is a $K_*$-line, and the HMS lemma assures us we have an invertible spectrum.

\[ \cdots \longrightarrow K_* M^0(p^i) \to K_* M^0(p^{i+1}) \longrightarrow \cdots \longrightarrow K_* M^0(p^\infty) \]

This suggests a way we can modify this construction: if we insert other maps which are $K_*$-homology isomorphisms, then we will not harm this proof that the colimit is an invertible spectrum. In particular, each spectrum $M^0(p^i)$ is type 1 and admits an Adams $v_i$-self-map $v_i^{p^j} : M^0(p^i) \to M^{-|v_i| p^j - 1}(p^i)$. With these maps in hand, select your favorite $p$-adic integer $a_\infty = \sum_{j=0}^\infty c_j p^j$ with $0 \leq c_j < p$, and construct the system:

\[ M^0(p) \to \cdots \to M^{-|v_i| p^{j-i}}(p^i) \to M^{-|v_i| p^{j-i+1}}(p^{i+1}) \to \cdots \to M^{-|v_i| p^{j-i+2}}(p^{i+2}) \to \cdots \to S^{-|v_i| a_\infty}. \]

Hopkins, Mahowald, and Sadofsky go on to check that this assignment $\mathbb{Z}^\wedge_p \to \text{Pic}_*$ is an injective, continuous homomorphism, and for $p > 2$ its cosets are represented by $S^0, \ldots, S^{-|v_i|}$. Unfortunately, this is the only sample family of exotic invertible spectra we know.\(^3\) To produce other examples, we at least now know what to look for: line bundles. Now, my favorite sort of vector bundles (for the purposes of this talk) are tangent bundles, and in particular when looking at a smooth point on a 1-dimensional variety, the stalk of the tangent bundle there is a 1-dimensional free module. There are certain spectra which chromatic homotopy theory tells us to think of in analogy to 1-dimensional formal affine lines, like $\Sigma^\infty_+ \mathbb{C}P^\infty$, and so one thing we could hope for is a dashed arrow completing the following commuting square:

\[ \begin{array}{c}
\{ \text{appropriate spectra} \} \\
\downarrow \; T_\eta \\
K_* \\
\downarrow \; T_\Sigma \\
\{ \text{formal affine varieties of dimension 1} \} \\
\downarrow \\
\text{VectorSpaces}_{K_*}
\end{array} \]

This is our goal for the rest of the talk.

\(^2\)This is a homotopical version of the statement that the $p$-primary part of the circle group $S^1$ is exactly the $p$-Prüfer group $\mathbb{Z}/p^\infty = \text{colim}_n \mathbb{Z}/p^n$.

\(^3\)This construction can be re-done using generalized Moore spectra to give a similar inclusion $\mathbb{Z}^\wedge_p \to \text{Pic}_*$ for $n > 1$.\[
2. Coalgebras

So, what we want is some spectrum-level operation which mimics the construction of the algebro-geometric tangent space. The primary obstacle to realizing this dream is a variance issue: classical algebraic geometry is done with rings, but we are concerned with covariant $K$-homology. So, while most typically investigate the formal scheme $X_K = \text{Spf} K^\ast X$, we should work out how to use $K X$ instead.

As a stepping stone, let's start by thinking about spaces $X$ such that $K^\ast X$ is finite as a $K^\ast$-module, i.e., $X_K$ is a finite scheme. In this case, $K X$ is simple to describe: it is the $K^\ast$-linear vector space dual of $K^\ast X$. Accordingly, where $K^\ast X$ had the structure of a $K^\ast$-algebra, $K^\ast X$ has the structure of a $K^\ast$-coalgebra. You can quickly check that the three categories of finite $K^\ast$-algebras, finite $K^\ast$-coalgebras, and finite schemes are all (co/contravariantly) equivalent. Diagrammatically, here are the functors available to us:

$$\begin{align*}
\text{Sch} & \xrightarrow{(-)\vee} \text{FiniteCoalgebras} \\
\text{Spec} & \xrightarrow{\emptyset} \text{FiniteAlgebras} \\
\text{Spec} & \xrightarrow{\emptyset} \text{FiniteSchemes}
\end{align*}$$

For a test ring $T$, the functor of interest $\text{Sch}$ is given by the formula

$$(\text{Sch} C)(T) = \left\{ u \in C \otimes T \mid \Delta u = u \otimes u \in (C \otimes T) \otimes_T (C \otimes T), \quad \varepsilon u = 1 \in T \right\}.$$

Now, let's approach the infinite case. It is no longer the case that we can get away with naively applying linear algebraic duality to turn algebras into coalgebras, because...

(1) ... the double-dual of an infinite-dimensional vector space is not isomorphic to the original.
(2) ... the inequality $(A \otimes A)^\vee \not\cong A^\vee \otimes A^\vee$ prevents us from dualizing the multiplication on a ring to a comultiplication on its dual.

Instead, there is a structure theorem for coalgebras which indicates what is happening: every finite dimensional vector subspace of a coalgebra can be finitely enlarged to a finite dimensional subcoalgebra — and, as a corollary, every coalgebra is ind-finite.

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In actuality, this ind-finiteness has been in the picture of algebraic-geometry-in-algebraic-topology all along: the formula $X_K = \text{Spf} K^\ast X$ means that we should consider the directed system of compact subspaces $X_s$ of $X$, then set $X_K = \text{colim}_s \text{Spec} K^\ast X_s$. This is what we mean when we say we're studying formal schemes: formal schemes are ind-finite schemes, meaning that they arise from pro-finite systems of rings or from ind-finite systems of coalgebras. And, in fact, $X_K$ can be modeled as $X_K = \text{Sch} K^\ast X$ for arbitrary $X$. This extends our diagram like so:

$$\begin{align*}
\text{Sch} & \xrightarrow{(-)\vee} \text{FiniteCoalgebras} \\
\text{Spec} & \xrightarrow{\emptyset} \text{FiniteAlgebras} \\
\text{Spec} & \xrightarrow{\emptyset} \text{FiniteSchemes} \\
\text{ProfiniteAlgebras} & \xrightarrow{\text{Spf}} \text{FormalSchemes}
\end{align*}$$

So, this new setting is very nearly just the old setting with one extra duality thrown in to the formula, and hence we can translate over any construction we want by dualizing appropriately. For a point $x : K^\ast X \to K^\ast$ in the affine scheme $X_K$, we would build the cotangent space by taking the corresponding ideal $I = \ker(K^\ast X \to K)$ and considering the quotient $T_s X_K = \text{coker}(I \otimes_{K^\ast X} I \to I)$. In the new setting, a point corresponds to a map $x : K_s \to K^\ast X$, from which we build the $(K_s X)$-comodule $M = \text{coker}(K_s \to K^\ast X)$, and then the tangent space

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4This is a pretty interesting case already, since it encompasses all manifolds, for instance.
Though it’s a low bar to clear, this middle category records much more information than bundles over a certain formal variety. It turns out that there is a refinement of the map \( \text{Pic} \). Our tangent space machine then immediately gives us some element of \( \text{Pic} \). This is all supposed to indicate what to do on the level of spectra: for an appropriate pointed coalgebra spectrum \( \eta \), the pullback square above is not sufficient, essentially because the fiber of spectra will also detect the cokernel of the tangent space given above — and hence \( X \) such spectrum is not a very interesting category: it has one isomorphism class, and it would be helpful to know more about it — in particular, it would be nice to know if it is also a standard sphere or something new.

One thing you may have noticed is that \( \text{Lines}_K \) is not a very interesting category: it has one isomorphism class, for instance. It turns out that there is a refinement of the map \( \text{Pic}_n \to \text{vector space} \), through the category of line bundles over a certain formal variety \( LT_n \), equivariant against the action of a certain \( p \)-adic analytic group \( \mathbb{S}_n \):

\[
\text{Pic}_n \overset{X}{\to} \text{line bundles}_{\mathbb{S}_n}(LT_n) \overset{\iota}{\to} \text{vector space}_{K}.
\]

Though it’s a low bar to clear, this middle category records much more information than \( \text{Lines}_K \). Hopf ring techniques allow us to identify the image of \( T_+ \Sigma^\infty_+ \mathbb{H}/p^\infty_{q-1} \) in this middle group: it is the determinantal bundle \( \Omega_{\mathbb{H}/p^\infty_{q-1}}^{n-1} \). This in particular means that it is not a standard sphere, and so we have uncovered a genuinely new invertible spectrum. Other authors have called similar spectra with this property “the determinantal sphere.”

Now that we’ve accomplished our original goal of producing an interesting invertible spectrum, I want to tell you about some other things that come out of these methods. The first, most basic point is that this construction can be iterated, yielding the following diagram of fiber sequences:

\[
\begin{array}{c}
I \otimes_A J & \longrightarrow & I \otimes A J \\
\downarrow & \alpha_1 \otimes 1 & \downarrow \\
I \otimes_k I & \longrightarrow & I \otimes_k A \otimes_k J.
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In the case of \( X = \Sigma_+^\infty \mathbb{CP}^\infty \), this recovers the cellular decomposition of \( \mathbb{CP}^\infty \). In the case of \( X = \Sigma_+^\infty \mathbb{HZ}/p^\infty_n \), this recovers something that feels identical to the cellular decomposition of \( \mathbb{CP}^\infty \), but where “cell” is extended to allow attaching maps along arbitrary elements of \( \text{Pic}_n \). In particular, one can show that the spectrum \( D_j \) is the \( j \)th smash power of \( D_1 \), as is the case for \( \mathbb{CP}^\infty \).

Given this striking similarity between \( \mathbb{CP}^\infty \) and \( \mathbb{HZ}/p^\infty_n \), one can ask what other things are lying around that are also similar. For instance, the map \( \mathbb{CP}^1 \to \mathbb{CP}^\infty \) is an element of homotopy called the Bott element \( \beta \), and we have an analogous map \( \beta : T_+ \Sigma_+^\infty \mathbb{HZ}/p^\infty_n \to \Sigma_+^\infty \mathbb{HZ}/p^\infty_n \). Moreover, Snaith showed that the spectrum \( \Sigma_+^\infty \mathbb{CP}^\infty [\beta^{-1}] \) is weakly equivalent to the \( K \)-theory spectrum \( KU \), and we can formally construct a Snaith-type determinantal \( K \)-theory spectrum \( R = \Sigma_+^\infty \mathbb{HZ}/p^\infty_n [\beta^{-1}] \). Westerland has shown a couple of remarkable features of this spectrum:

1. \( R \) is weakly equivalent to the fixed point spectrum \( E^hS_{S_n} \), where \( S_{S_n} \) is the subgroup of “special” elements of the stabilizer group, i.e., those in the kernel of the determinant.
2. The space \( \Omega^\infty R(1) \) supports an analogous map to \( BGL_1 S \), yielding a determinantal theory of Thom spectra. In particular, there is a Thom spectrum analogous to \( MU \), which classifies those spectra which are appropriately oriented against \( \mathbb{HZ}/p^\infty_n \).
3. ...There is even an analogue of the image of \( J \)...

Westerland answers a lot of questions and opens a lot of doors, and there are a lot of things left to investigate. Though there are many others, here are two yet-unanswered questions I’d like to leave you with:

1. Analogues of the spaces \( \Omega SU(m) \) can be constructed, but analogues of the spaces \( BU(m) \) are unknown. What should the analogue of a rank \( m \) bundle be, where \( \mathbb{HZ}/p^\infty_n \) classifies “rank 1 bundles”?
2. The tangent space analysis also applies 2-adically to \( \mathbb{RP}^\infty \), where it’s found that \( T_+ \Sigma_+^\infty \mathbb{RP}^\infty \simeq S^1 \). In both the real and complex cases, it is the case that the tangent space agrees with \( \Sigma O(1) \) and \( \Sigma U(1) \) respectively — that’s because these spaces can be realized as bar constructions. Does \( \Sigma^{-1} T_+ \Sigma_+^\infty \mathbb{HZ}/p^\infty_n \) have an interpretation as a base case of a Lie group? Does it have an unstable realization, as \( O(1) \) and \( U(1) \) do? How does the annular tower compare to the bar filtration?