The Distortion of a Knotted Curve

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(Dated: August 20, 2004; revised September 30, 2006)

The distortion of a curve measures the maximum arc/chord length ratio. Gromov showed any closed curve has distortion at least $\pi/2$ and asked about the distortion of knots. Here, we use the existence of an essential secant to show that any nontrivial knot in space has distortion at least 3.9945; examples show that distortion under 8.2 suffices to build a trefoil knot.

Gromov introduced the notion of distortion for curves as the maximum ratio of arclength to chord length. (See [Gro78], [Gro83, p. 114] and [GLP81, pp. 6–9].) He showed that any closed curve has distortion at least $\pi/2$, that of a round circle. (See also [KS97].) He then asked whether every knot type can be built with distortion less than, say, 100.

Mullikin [Mul06] has run some preliminary numerical experiments looking for knots of low distortion. He has found, for instance, that a trefoil can be built with distortion less than 7.16. An open trefoil (a long knot with straight ends) must evidently have somewhat greater distortion, but again an example shows that this can be built with distortion less than 9.3. Then connect sums of arbitrarily many trefoil knots can be built with this same distortion (under 9.3). Indeed, by rescaling the summands smaller and smaller, we can even produce a wild knot—an infinite connect sum—with this same distortion, as sketched in Figure 1.

Although infinite families of knots as above can be built with uniformly bounded distortion, many people expect this cannot be true for all knots, giving a negative answer to Gromov's question above. Litherland *et al.* proved [LSDR99] that distortion is bounded above by half the ropelength of a knot, but this does not help answer Gromov's question. We provide a first step towards understanding the distortion of knots with a bound from below; we prove that any (nontrivial tame) knotted curve has distortion at least 3.9945, more than twice what is possible for an unknotted closed curve.

Independent recent work in computational geometry [EBGK04a, EBGK04b, DGR04] has considered distortion (under the name "geometric dilation") for curves—or more generally graphs—in the plane. In particular, [EBGK04b] constructs a graph of distortion less than 1.678 which (up to Euclidean similarity) can cover any finite point set. One result of [DGR04] says that a closed plane curve with distortion close to $\pi/2$ must be C^0-close to a round circle. We note below how that argument can be modified to apply to closed curves in higher ambient dimensions. But, of course, being C^0-close to an unknot does not preclude being knotted.

Our main results are based on a quite different line of argument. When studying quadrisecants of knots—lines in space that intersect the knot four times—Kuperberg [Kup94] introduced a way to say which secants of the knot are topologically nontrivial or *essential*. Denne [Den04] has further developed these ideas to show that knotted curves have essential alternating quadrisecants, and with Diao [DDS06], we used such quadrisecants to get a good lower bound on the ropelength of nontrivial knots. Here, to obtain new bounds for distortion, we merely use the existence of an essential secant, along with results from [DDS06] that characterize how a family of secants can become essential.

Perhaps one can use the existence of an essential quadrisecant to get even better bounds. However, distortion is an infamously "slippery" notion: the distortion for any particular pair of points along a knot can often be decreased towards 1 simply by an affine stretch of the knot. Thus it is not clear how to use a projectively invariant notion—like the existence of a quadrisecant—to bound distortion.

1. BASIC RESULTS ON DISTORTION

We will deal throughout with oriented rectifiable curves γ embedded in $\mathbb{R}^n$ with finite length $\ell(\gamma)$. Such a curve γ has a Lipschitz parameterization by arclength, defined either on $[0,\ell(\gamma)]$ if γ is an *arc* or on $\mathbb{R}/\ell(\gamma)\mathbb{Z}$ if γ is a *knot*, a simple closed curve. (As our choice of nomenclature indicates, we are mainly interested in the case $n = 3$.)

Two points p, q along a knot K separate K into two complementary arcs, γ_{pq} and γ_{qp}. (Here γ_{pq} is the arc from p to q following the orientation of K.) We let ℓ_{pq} denote the length of γ_{pq}. We are mainly interested in the shorter arclength distance $d(p,q) := \min(\ell_{pq},\ell_{qp}) \leq \ell(\gamma)/2$. We contrast this with the straight-line (chord) distance $|p-q|$, the length of the segment $\overline{pq} \subset \mathbb{R}^n$. Given any point p on a knot K, there is a unique *opposite point* p^* such that $\ell_{pp^*} = \ell(K)/2$. (The situation is simpler if we start with an arc γ: if p, q are two points in order along γ, then $d(q,p) = d(p,q) := \ell_{pq}$ is the length of the subarc γ_{pq}.)

The arclength parametrization of γ has Lipschitz constant 1

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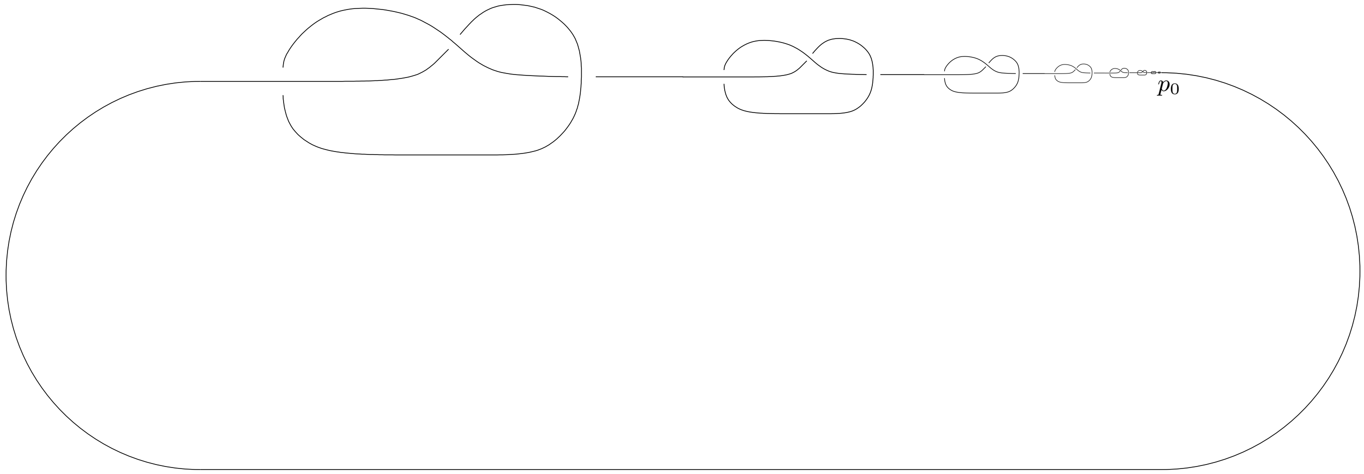

Figure 1: A wild knot, the connect sum of infinitely many trefoils, can be built with distortion less than 10.7, simply repeating infinitely many scaled copies of a disortion-minimizing open trefoil knot. We merely need to ensure that the copies are sufficiently small compared to the overall loop of the knot and sufficiently distant from each other (perhaps slightly more than in this sketch) to know that the maximum distortion will be realized within one trefoil. This knot has a single point p_0 where it fails to be a smooth curve.

by definition. The distortion of γ is a Lipschitz constant for the inverse map:

Definition. The *distortion* between distinct points p and q on the curve γ is

$$\delta(p,q) := \frac{d(p,q)}{|p-q|} \geq 1.$$

The distortion of γ is the maximum distortion between any points:

$$\delta(\gamma) := \sup_{p,q} \delta(p,q),$$

where the supremum is over the set $(\gamma \times \gamma) \setminus \Delta$ of all pairs of distinct points. On a closed curve K, we can also consider a restricted distortion, only considering opposite pairs:

$$\delta_\circ(K) := \max_{p \in K} \delta(p,p^*).$$

Clearly $\delta_\circ(K) \leq \delta(K)$ since the supremum is over a subset.

For any knot K, we have $\delta_\circ \geq \pi/2$, with equality only for a round circle. A proof following Gromov can be found in [KS97, Prop. 2.1]. Independently, [DGR04] used similar arguments to bound the shape of closed plane curves with distortion not much more than $\pi/2$. Although the proof does not carry over directly to higher dimensions, below in Section 4 we make the necessary modifications to get a similar result: a knot K with distortion close to $\pi/2$ must be C^0-close to a round circle. Of course, this says nothing about the knot type of K: indeed, any knot can be realized with $\delta_\circ$ arbitrarily close to $\pi/2$. Our main result, on the other hand, says that a nontrivial knot must have overall distortion at least 3.9945.

We pause to give a much shorter proof of the observation [LSDR99] that distortion is bounded above by half the ropelength. The ropelength $R(K)$ of a knotted curve K is the (scale invariant) quotient of length over thickness. The thickness $\tau(K)$ of a space curve is defined [GM99] to be twice the infimal radius $r(x,y,z)$ of a circle passing through any three distinct points of K. A link is $C^{1,1}$ (that is C^1 with Lipschitz tangent vector) if and only if it has positive thickness [CKS02]. Of course when K is C^1, we can define normal tubes about K and then $\tau(K)$ is the supremal diameter of such a tube that remains embedded. It is straightforward to show that the ropelength of any knot is at least π.

Proposition 1.1. *The distortion of any tame knot K is bounded above by $R(K)/2$.*

Proof. If K has zero thickness then $\delta(K) \leq R(K)/2 = \infty$. Assume K has unit thickness. Given $p, q \in K$ first assume that $|p-q| \leq 1$. Standard results on the geometry of thick curves ([DDS06] Lemma 3.1) show that the arclength of K between p and q is at most $\arcsin(2|p-q|)$. Hence

$$\delta(p,q) \leq \frac{\arcsin(2|p-q|)}{|p-q|} \leq \frac{\pi}{2} \leq \frac{R(K)}{2}.$$

Now assume $|p-q| \geq 1$, then the arclength of K between p and q is at most $R(K)/2$. Thus for any $p, q \in K$, $\delta(p,q) \leq R(K)/2$. □

In this paper, we do not investigate criticality conditions for distortion. Perhaps techniques like those of [CFK$^+$04] could be used to develop an analogous balance criterion for curves whose distortion cannot be decreased by small motions; but there seem to be extra technical difficultites here. We do note, as mentioned in [KS97], that if p and q are a pair of nonopposite points realizing the distortion $\delta(K)$ and if K has a well-defined tangent direction T_p at p, then the angle between T_p

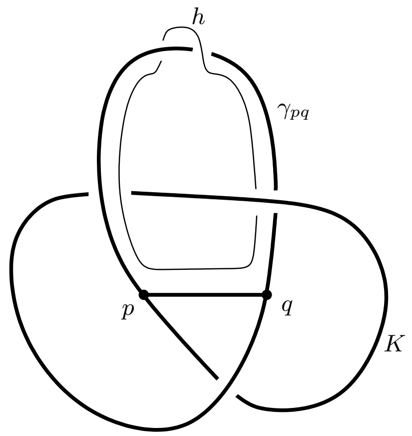

Figure 2: The arc γ_{pq} is essential in the knot K because the parallel $h(\gamma_{pq}, \overline{pq}, \gamma_{qp})$, whose linking number with K is zero, is homotopically nontrivial. In this example, γ_{qp} is also essential, so pq is essential.

and $\overrightarrow{qp}$ must be arcsecδ. Now suppose we have an arc with constant distortion from a fixed point p; the angle condition above says the arc is part of a spiral:

Example. Consider a logarithmic spiral S_ψ making constant angle ψ with circles around its center point. It can be parametrized as $S_\psi(t) := e^{t\tan\psi}(\cos t, \sin t)$, with speed $e^{t\tan\psi}\sec\psi$ for all $t \in \mathbb{R}$. A full turn of this spiral, say the arc $S_\phi([0, 2\pi])$, realizes the distortion $\delta = \csc\psi$: its arclength is $\csc\psi(e^{2\pi\tan\psi} - 1)$ while its endpoints are at distance $e^{2\pi\tan\psi} - 1$. If we draw any ray out from the limit point $q = (0, 0) = S_\psi(-\infty)$ of the sprial, it intersects S_ψ infinitely often, and the the distortion between any two of these intersection points is this same δ, as is the distortion from q to any point on S_ψ.

2. ESSENTIAL SECANTS AND DISTORTION

To get a good lower bound for the distortion of nontrivial knots we use the notion of essential arcs, introduced in [DDS06] as an extension of ideas of Kuperberg [Kup94]. Note that generically a knot K together with a chord $\overline{pq}$ forms a Θ-graph in space; being essential is a topological feature of this knotted graph, as shown in Figure 2.

Definition ([DDS06]). Suppose α, β and γ are three interior-disjoint arcs from p to q, forming a knotted Θ-graph in $\mathbb{R}^3$. Given an ordering (α, β, γ) of these arcs, we define a loop $h = h(\alpha, \beta, \gamma)$ in the free homotopy of the knot complement $X := \mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$. Namely, h is represented by a parallel curve to $\alpha \cup \beta$, chosen to have linking number zero with $\alpha \cup \gamma$ (that is, chosen to be trivial in the homology of X). Then we say the ordered triple (α, β, γ) is *essential* if $h(\alpha, \beta, \gamma)$ is a nontrivial free homotopy class, (or equivalently, if $\alpha \cup \beta$ bounds no disk whose interior is disjoint from $\alpha \cup \gamma$).

Now suppose K is a knot and $p, q \in K$. Assuming $\overline{pq}$ has no interior intersections with K, we say γ_{pq} is an *essential* arc of K if $(\gamma_{pq}, \overline{pq}, \gamma_{qp})$ is essential. If $\overline{pq}$ does intersect K, we say γ_{pq} is *essential* if for any $\varepsilon > 0$ there is an ε-perturbation S of $\overline{pq}$ such that $(\gamma_{pq}, S, \gamma_{qp})$ is essential.

Note that the last part of the definition ensures that, within the set $(K \times K) \smallsetminus \Delta$ of all subarcs, the set of essential arcs is closed. If K is unknotted then any subarc is inessential [CKKS03]. Conversely, Dehn's lemma can be used to show the following:

Lemma 2.1 ([DDS06], Thm. 5.2). *If for some $p, q \in K$, both γ_{pq} and γ_{qp} are inessential, then K is unknotted.* □

Definition. Given $p, q \in K$, we say the secant pq is *essential* if both arcs γ_{pq} and γ_{qp} are essential.

What will be most important for us is the following theorem which describes borderline-essential arcs.

Theorem 2.2 ([DDS06], Thm. 7.1). *Suppose γ_{pr} is in the boundary of the set of essential arcs for a knot K. (That is, γ_{pr} is essential, but there are inessential arcs of K with endpoints arbitrarily close to p and r.) Then K must intersect the interior of segment $\overline{pr}$ at some point $q \subset \gamma_{rp}$ for which the secants pq and qr are both essential.* □

We also quote an elementary geometric lemma about minimum-length arcs avoiding a unit ball in space. (A two-dimensional version can be dated back to [Kub23], as noted in [EBGK04a, DGR04].)

Definition. For $r \geq 1$, let $f(r) := \sqrt{r^2 - 1} + \arcsin(1/r)$. For $r, s \geq 1$ and $\theta \in [0, \pi]$, the minimum length function is defined by

$$m(r, s, \theta) := \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos\theta} & \text{if } \theta \leq \theta_0(r, s) \\ f(r) + f(s) + (\theta - \pi) & \text{if } \theta \geq \theta_0(r, s) \end{cases},$$

where $\theta_0(r, s) = \arccos(1/r) + \arccos(1/s)$.

Lemma 2.3 ([DDS06], Lem. 4.3). *Any arc $\gamma \subset \mathbb{R}^n$ staying outside $B_1(p)$ has length at least*

$$m\big(|a - p|, |b - p|, \angle apb\big),$$

where a and b are the endpoints of γ. □

Note that $m(r, s, \theta)$ is not always increasing in r; it is decreasing when $r < s\cos\theta$ and $\theta < \theta_0$. If we define

$$n(s, \theta) := \min_{r \geq 1} m(r, s, \theta),$$

a straightforward computation gives

$$n(s, \theta) = \begin{cases} s\sin\theta & \text{if } \theta \leq \arccos(1/s) \\ f(s) + \theta - \pi/2 & \text{if } \theta \geq \arccos(1/s) \end{cases}.$$

This function is now increasing in s and in θ, and we have

Corollary 2.4. *Any arc γ from a to b staying outside $B_1(p)$ has length at least $n\big(|b-p|, \angle apb\big)$.* □

As we note below, the monotonicity of n would now be enough to get a distortion bound of $2n(3/2, \pi/2) > 3.695$. To get our somewhat better bound, we now consider the minimum length of a somewhat more restricted class of curves avoiding a ball.

For any angle $\psi \le \pi/2$, define the continuous function

$$m_\psi(\phi) := \begin{cases} \sin\phi/(1-\sin\phi\sin\psi) & \text{if } \phi \le \psi \\ \big(e^{(\phi-\psi)\tan\psi}\sec^2\psi - 1\big)/\sin\psi & \text{if } \phi \ge \psi \end{cases}$$

and note that it is increasing in $\phi \in [0, \pi]$. The meaning of this, as the length of a particular spiral curve, will become clear in the course of proving the proposition below.

Proposition 2.5. *Given $\psi \le \pi/2$, suppose $\gamma_{ab} \subset \mathbb{R}^n \smallsetminus B_1(q)$ is an arc with the property that for all $x \in \gamma$ we have $|x-q| \ge 1 + \ell_{ax}\sin\psi$. Then $\ell_{ab} \ge m_\psi(\angle aqb)$.*

Proof. Set up polar coordinates (ρ, θ) centered at q with a on the ray $\theta = 0$. Fixing ψ and $\phi := \angle aqb$, consider the class $\mathcal{C}$ of arcs starting at $\theta = 0$ and ending at $\theta = \phi$, parametrized by arclength s, with

$$\rho(s) \ge 1 + s\sin\psi. \tag{1}$$

We first describe an arc $\gamma_0 \in \mathcal{C}$ whose length is $m_\phi(\psi)$. Then we will show that $\mathcal{C}$ has a shortest element and that it must be this same γ_0.

For $\phi \ge \psi$, the arc γ_0 is the C^1 join of two pieces, as shown in Figure 3: The first is a vertical segment $\overline{ab}$ given by $\rho = \sec\psi\sec\theta$ for $0 \le \theta \le \psi$. The fact that the segment subtends angle ψ at q means that it can be joined in a C^1 fashion at b to a spiral S_ψ. The radial distance $|a-q| = \sec\psi$ is then chosen so that (1) holds with equality at b. The second piece is then the spiral S_ψ given by $\rho = e^{(\theta-\psi)\tan\psi}\sec^2\psi$ for $\psi \le \theta \le \phi$. Straightforward calculations show that the length of γ_0 is

$$m_\psi(\phi) = \big(e^{(\phi-\psi)\tan\psi}\sec^2\psi - 1\big)/\sin\psi.$$

For $\phi \le \psi$, the arc γ_0 is simply the vertical segment $\rho = \sec\theta\cos\phi/(1-\sin\phi\sin\psi)$. Its radial distance

$$|a-q| = \cos\phi/(1-\sin\phi\sin\psi) \ge \sec\phi$$

is again chosen to give equality in (1) at the point $b = c$; the segment would not join a spiral S_ψ in a C^1 fashion, but this is irrelevant in this case since we have no spiral. The function $m_\psi(\phi)$ was defined in the case $\phi \le \psi$ to be the length $\sin\phi/(1-\sin\phi\sin\psi)$ of this segment.

Let $\mathcal{C}_0 \subset \mathcal{C}$ be the subclass of all curves with length at most $m_\psi(\phi)$. The curve γ_0 shows that this is nonempty. By (1) no curve in $\mathcal{C}_0$ gets further from q than γ_0 does. Thus standard compactness results for spaces of Lipschitz functions show that $\mathcal{C}_0$ is compact: any sequence has a subsequence converging in sup norm to a Lipschitz curve. Even if arclength drops in the limit, (1) is still satisfied everywhere.

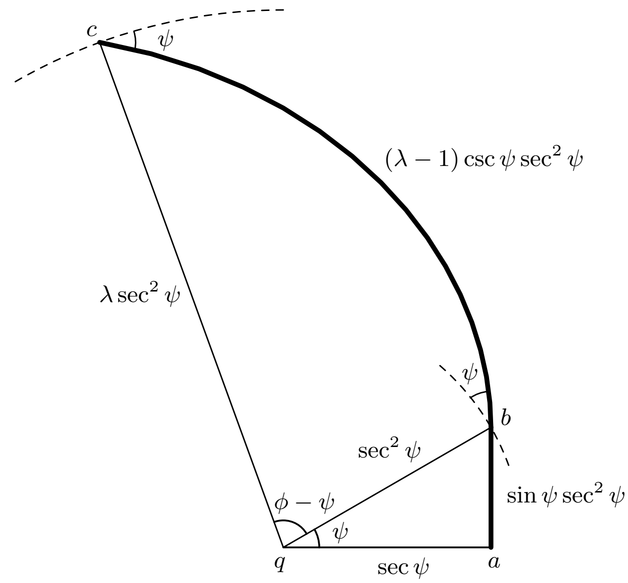

Figure 3: This figure shows the shortest curve γ_0 guaranteed by Proposition 2.5. In labeling the lengths, we use the abbeviation $\lambda := e^{(\phi-\psi)\tan\psi}$. The curve consists of a vertical segment $\overline{ab}$ joined in a C^1 fashion to a spiral S_ψ making constant angle ψ with the dashed circles centered at q.

Compactness shows that $\mathcal{C}_0$ has a curve γ_1 of minimum length. We will use the regularity of this minimizer to prove that it is γ_0. Note first that strict inequality in (1) holds for some open set O of values s. On the closure of any interval in O, the minimizer γ_1 must be straight, since otherwise its length could be decreased without violating (1). At the endpoints of such an interval where (1) holds with equality, the segment must make angle ψ with circles around q (that is, must be tangent to a spiral S_ψ): if it went inwards it would violate (1); if it went outwards the curve could be shortened by pulling its endpoint outwards.

Next we note that along any line the angle it makes with circles around q increases monotonically; so the line is tangent only once to such a spiral S_ψ. It follows that the minimizer γ_1 consists of at most three pieces: a segment, a sprial arc, and another segment.

Now consider the endpoints of γ_1, given the condition that they are at $\theta = 0$ and $\theta = \phi$. If (1) is strict at either endpoint, then γ_1 must meet the radial line from q perpendicularly there. If (1) holds with equality, then we only get a one-sided inequality for this angle. We conclude that at the starting point a, there must be strict inequality; at the final point b there must be equality. This determines the geometry uniquely: $\gamma_1 = \gamma_0$, with length $m_\psi(\phi)$ as claimed. □

Our main theorem is now based on analysis of the shortest essential arc in a knot K and the two related essential secants guaranteed by Theorem 2.2. We delay until Section 3 below the proof that rectifiable nontrivial tame knots always have a shortest essential arc. Note, however, that a topologically wild knot, even if its distortion is low, might have arbitrarily short essential arcs, as in Figure 1; our theorem does not apply to such knots.

Theorem 2.6. *Every nontrivial tame knot has distortion $\delta > 3.9945$.*

Proof. Non-rectifiable curves have infinite distortion. By Theorem 3.4 a rectifiable nontrivial tame knot K has a shortest essential arc which we denote γ_{pr}. Let δ be the distortion of K. For convenience, rescale the knot so that $\ell_{pr} = \delta$. Then any essential secant ab has $|a - b| \geq 1$, for otherwise the shorter of the essential arcs γ_{ab} and γ_{ba} would have length at most $|a - b|\delta < \delta$, contradicting the definition of γ_{pr}. Note that by Theorem 2.2 it follows that $|a - b| \geq 2$ whenever γ_{ab} is borderline-essential.

Since some nearby arcs are shorter and thus inessential, Theorem 2.2 is applicable to γ_{pr}, giving us $q \in \overline{pr} \cap \gamma_{rp}$ with pq and qr essential. Now let m be the midpoint of γ_{pr}, so that $\ell_{pm} = \delta/2 = \ell_{mr}$. If follows that for any $x \in \gamma_{pm}$ we have $d(q, x) \geq \delta + \ell_{px}$, while for $y \in \gamma_{mr}$ we have $d(q, y) \geq \delta + \ell_{yr}$. By the definition of distortion, if follows that $|q - x| \geq 1 + \ell_{px}/\delta \geq 1$. In particular, $|q - m| \geq 3/2$, and the whole arc γ_{pr} stays outside $B_1(q)$. It follows immediately that $\delta = \ell_{pr} \geq \pi$, which is twice the minimum distortion possible for a closed unknotted curve.

Now let $\alpha := \angle pqm$ and $\beta := \angle mqr$. We pause to derive the bound $\delta \geq 3.695$ since it illustrates how the full argument will work. By Corollary 2.4, we have

$$\ell_{pm} = \ell_{mr} \geq \max\Big(n\big(\tfrac{3}{2}, \alpha\big), n\big(\tfrac{3}{2}, \beta\big)\Big).$$

But $\alpha + \beta = \pi$ since pqr are collinear. Thus

$$\begin{aligned}\delta &= \ell_{pm} + \ell_{mr} \geq n\big(\tfrac{3}{2}, \max(\alpha, \beta)\big)\\ &\geq 2n\big(\tfrac{3}{2}, \tfrac{\pi}{2}\big) = 2f\big(\tfrac{3}{2}\big) = \sqrt{5} + 2\arcsin\big(\tfrac{2}{3}\big)\\ &> 3.695.\end{aligned}$$

To get the somewhat better bound of the theorem, we apply Proposition 2.5. Choosing $\psi \in [0, \pi/2]$ so $\csc\psi = \delta$, we have its hypotheses satisfied for γ_{pm}; thus $\ell_{pm} \geq m_\psi(\alpha)$. We can apply the proposition also to γ_{mq} after reversing its orientation, giving $\ell_{mq} \geq m_\psi(\beta)$. Remembering that m_ψ is an increasing function, since m is the midpoint we have

$$\begin{aligned}\csc\psi = \delta = \ell_{pm} + \ell_{mr} &\geq 2m_\psi(\max(\alpha, \beta))\\ &\geq 2m_\psi(\pi/2) = 2\big(e^{(\pi/2-\psi)\tan\psi}\sec^2\psi - 1\big)\csc\psi.\end{aligned}$$

Asking for equality here gives the equation $3\cos^2\psi = e^{(\pi/2-\psi)\tan\psi}$, with a unique solution $\psi_0 \approx 0.253035$ corresponding to $\delta_0 \approx 3.99451$. To satisfy the inequality we then need $\delta \geq \delta_0 > 3.9945$. □

We note that this bound is not sharp. In particular, if γ_{pq} and γ_{qr} were really also shortest essential arcs, their geometry would have to be like that of γ_{pr} and in particular their endpoints would be relatively far apart. We have, however, resisted the temptation to pursue this argument even far enough to improve our lower bound to 4.

3. TAME KNOTS HAVE A SHORTEST ESSENTIAL ARC

In our work on ropelength, we showed that short arcs of thick knots are inessential. In particular, for a knot of unit thickness, we showed [DDS06, Lemma 8.1] that essential arcs have length at least π, and that if $\overline{pq}$ is essential then $|p - q| \geq 1$. Here, we show that sufficiently short arcs of any rectifiable tame knot are inessential.

Lemma 3.1. *Suppose B is a round ball and K is a tame knot such that $K \cap B$ is a diameter of the ball. Given two points $p, q \in K \cap B$ let γ_{pq} denote the subarc of K from p to q entirely inside B. Let β be an arc within B from p to q disjoint from K. Then $(\gamma_{pq}, \beta, \gamma_{qp})$ is inessential.*

Proof. Pick any homeomorphism between γ_{pq} and β. Join all pairs of corresponding points by straight segments; these fill out a disk with boundary $\gamma_{pq} \cup \beta$, which by convexity stays entirely within B. The disk avoids K (except of course for the segment endpoints along γ_{pq}) because β avoids the straight segment $B \cap K$. □

Proposition 3.2. *Given any point a on a tame knot K, there is some $r > 0$ such that, for any $p, q \in K \cap B_r(a)$ the secant $\overline{pq}$ is inessential, and in particular, if the arc γ_{pq} is contained in $B_r(a)$ then this arc is inessential.*

Proof. Because K is tame, given any point $a \in K$, we can find an isotopy I taking K to a polygonal knot such that that $I(a)$ is not a vertex. Then around $I(a)$ is some round ball B_0 such that $I(K) \cap B_0$ is a diameter as in the statement of Lemma 3.1. Then $U := I^{-1}(B_0)$ is a neighborhood of $a \in K$. (In fact, $(U, U \cap K)$ is an unknotted ball-arc pair as in the definition of locally flat.)

Since it is an open neighborhood, U must contain a round ball B centered at a. Let p, q be any two points in $K \cap B$. (Note that $K \cap B$ is not necessarily a single arc.) The segment $\overline{pq}$ is contained in B by convexity, hence in U. So is any sufficiently small perturbation S of this segment (as in the definition of essential). Let γ_{pq} be that subarc of K from p to q which is entirely contained in U (though perhaps not in B).

Now apply the isotopy I again: $\beta := I(S)$ is an arc from $I(p)$ to $I(q)$ within $B_0 = I(U)$. By Lemma 3.1,

$(I(\gamma_{pq}), \beta, I(\gamma_{qp}))$ is inessential. Applying I^{-1} again, we see that γ_{pq} is inessential for K, hence that $\overline{pq}$ is inessential. □

Theorem 3.3. *Given any tame knot K, there is some $\varepsilon > 0$ such that, for any $p, q \in K$ with $|p - q| < \varepsilon$, the secant $\overline{pq}$ is inessential.*

Proof. The Euclidean metric on $\mathbb{R}^3$ restricts to a metric on K (the chord metric). We can use the l_∞ combination of this metric with itself as a metric on $K \times K$. Then Proposition 3.2 shows exactly that for each $a \in K$, there is some r such that for all (p, q) in the r-ball around $(a, a) \in K \times K$, the secant $\overline{pq}$ is inessential.

These balls cover a neighborhood of the diagonal $\Delta \subset K \times K$. Since Δ is compact, the neighborhood must include an ε-neighborhood for some $\varepsilon > 0$. But if $|p - q| < \varepsilon$ then (p, q) is distance less than ε from $(p, p) \in \Delta$. □

Theorem 3.4. *Any nontrivial, tame, rectifiable knot K has a shortest essential subarc.*

Proof. Since K is rectifiable, length is a well-defined function on subarcs. Consider the set of subarcs of K of length at most half the length of K. Identifying the subarc γ_{pq} with (p, q), this can be viewed as a half-open annulus $A \subset K \times K \smallsetminus \Delta$. The closure of A is $A \cup \Delta$, including pairs on the diagonal corresponding to zero-length subarcs. The subset $E \subset A$ consisting of essential subarcs is, by definition of essential, (relatively) closed. By Proposition 3.2, E avoids some r-neighborhood of each point $(a, a) \in \Delta$, so E is in fact closed in the compact set $A \cup \Delta$, meaning that E itself is compact.

If two complementary subarcs of a knot are both inessential, [DDS06, Thm. 5.2] shows by Dehn's lemma that the knot is trivial. Thus the fact that K is nontrivial implies that E is nonempty. The length functional thus achieves a minimum on the nonempty compact set E. □

4. CURVES WITH SMALL DISTORTION BETWEEN OPPOSITE POINTS

Gromov showed that the only closed curve with distortion exactly $\pi/2$ is the round circle. (See [KS97, Prop. 2.1].) For plane curves, Dumitrescu et al. [DGR04] have quantified a relation showing that a closed curve of distortion not much more than $\pi/2$ must be close to a round circle. Here, we build on both these techniques in order to get a similar result valid in arbitrary dimensions.

More precisely, our result considers only the distortion $\delta_\circ$ between opposite points on the curve; we show that any knot K with $\delta_\circ^2 \le \pi^2/4 + \varepsilon^2$ must be 2ε-close to a round circle in the C^0 sense. There is no converse result: one can find knots K arbitrarily close to a circle—but zigzagging to have much greater arclength—for which $\delta_\circ$ is as large as desired. We note, however, that a curve which is C^1-close to a circle has not only $\delta_\circ$ but also δ close to $\pi/2$, since distortion is continuous in the C^1 topology.

We also note that there is no analog to our main theorem for $\delta_\circ$. When $\delta_\circ(K)$ is small, K is C^0-close to a circle, but this does not prevent K from being knotted. Indeed, any knot type can be built as a small local knot in a huge round circle, and then it has $\delta \gg \delta_\circ \approx \pi/2$.

Our first lemma is essentially based on [DGR04, Lem. 2]:

Lemma 4.1. *Suppose $\alpha(t)$ and $\beta(t)$ are two Lipschitz curves in $\mathbb{R}^n$ with perpendicular velocity vectors $\langle \dot\alpha(t), \dot\beta(t) \rangle = 0$ almost everywhere. Then the length of the curve $\gamma(t) := \alpha(t) + \beta(t)$ satisfies*

$$\sqrt{(\ell(\alpha))^2 + (\ell(\beta))^2} \le \ell(\gamma) \le \ell(\alpha) + \ell(\beta).$$

Proof. Define a plane curve starting at $(0, 0)$ and with velocity vector $\big(|\dot\alpha(t)|, |\dot\beta(t)|\big)$. This curve has the same length as γ. It proceeds monotonically up and to the right, ending at the point $\big(\ell(\alpha), \ell(\beta)\big)$. The desired inequalities follow from plane geometry. □

Proposition 4.2. *Suppose $K \subset \mathbb{R}^n$ is a closed curve with $\delta_\circ(K) \le \sqrt{\pi^2/4 + \varepsilon^2}$, for $\varepsilon < 1/2$. Then (up to Euclidean similarity) K is within distance 2ε of the unit circle in the C^0 sense. In particular K lies (homotopic to the core) within a 2ε-tube around the unit circle.*

Proof. Define $\delta := \delta_\circ(K)$, rescale so that $\ell(K) = 4\delta$, and parameterize the curve by arclength $s \in \mathbb{R}/4\delta\mathbb{Z}$. Opposite points $p = K(s)$ and $p^* = K(x + 2\delta)$ have arclength $d(p, p^*) = 2\delta$, so we have $|p - p^*| \ge 2$.

As in [KS97, DGR04] we can centrally symmetrize K while only decreasing $\delta_\circ$. Define new curves

$$\gamma^\pm(s) := (p \pm p^*)/2 = \big(K(s) \pm K(s + 2\delta)\big)/2$$

in $\mathbb{R}^n$. Although they are not arclength-parametrized, each is Lipschitz with speed at most one. (Our curves are rectifiable and thus have tangent vectors almost everywhere; by the triangle inequality $2|\dot\gamma^\pm(s)| \le |\dot K(s)| + |\dot K(s + 2\delta)| = 2$.) The curve γ^- is centrally symmetric, and because of our rescaling above, it lies outside the unit sphere in $\mathbb{R}^n$, touching it at least at some pair of antipodal points. As for any such symmetric curve, we then have $\delta_\circ(\gamma^-) = \ell(\gamma^-)/4$.

Following [DGR04], the length of γ^+ can be bounded in terms of the difference between the distortions of γ^- and K. In particular K can be recovered as the sum $K = \gamma^- + \gamma^+$, and the tangent vectors (being the sum and difference of unit vectors) are perpendicular: $\langle \dot\gamma^+, \dot\gamma^- \rangle \equiv 0$. It follows by Lemma 4.1 that

$$\ell(\gamma^+)^2 + \ell(\gamma^-)^2 \le \ell(K)^2.$$

Next we note that γ^+ in fact double-covers a closed curve of half its length. Now any closed curve of length ℓ lies within some ball of radius $\ell/4$. Combining these facts, we find that γ^+ lies within a ball of radius

$$r_1 := \tfrac{1}{2}\sqrt{\delta^2 - \delta_\circ(\gamma^-)^2} \le \varepsilon/2.$$

The inequality follows from our hypothesis on δ since γ^-, being a closed curve, has $\delta_\circ \ge \pi/2$. Of course since $K = \gamma^- + \gamma^+$, this r_1 is an estimate for the C^0 distance from K to γ^-.

We note at this point that following the argument of [DGR04] further would show that the curve K of small distortion lies in a thin spherical shell around the unit sphere. While in $\mathbb{R}^2$ that essentially implies the desired result, in higher dimensions we must work a bit harder.

So now let $\hat\gamma$ be the radial projection of γ^- inwards to the unit sphere. Since this is again a centrally symmetric curve of minimum radius 1, we have $\ell(\hat\gamma) = 4\delta_\circ(\hat\gamma)$. Define $r_2 := \sup|\gamma^-| - 1$ and a curve $\tilde\gamma$ by $\gamma^- = \hat\gamma + \tilde\gamma$. Then $\hat\gamma$ and $\tilde\gamma$ again have perpendicular velocities (tangent and normal to the unit sphere, respectively). The centrally symmetric $\tilde\gamma$ lies in the ball of radius r_2 by definition, but no smaller ball. Thus its length is at least $4r_2$. Applying Lemma 4.1 again, we find

$$r_2 \le \sqrt{\delta_\circ(\gamma^-)^2 - \delta_\circ(\hat\gamma)^2} \le \varepsilon,$$

where r_2 is the C^0 distance from γ^- to $\hat\gamma$.

We have now reduced to the case of a centrally symmetric curve $\hat\gamma$ on the unit sphere, parametrized with speed at most one on the interval $[0, 4\delta]$, which is only slightly longer than 2π. We claim it is C^0-close to an equator of the sphere.

Consider the function $f(t) := \langle\hat\gamma(t), \hat\gamma(t+\delta)\rangle$. Because $\hat\gamma$ is centrally symmetric, $f(t) = -f(t+\delta)$, so f is zero somewhere. That is, we have four points $\pm a$, $\pm b$, equally spaced with respect to the parametrization of $\hat\gamma$, with $a \perp b$. Our claim now in particular is that $\hat\gamma$ is close to the great circle through these points. It suffices to compare the subarc γ_{ab} to the geodesic arc α from a to b parametrized at constant speed $\pi/2\delta$ over the same interval $t \in [0, \delta]$.

Now suppose the C^0 distance from γ_{ab} to α is r_3. That is, for some t we have $|\gamma_{ab}(t) - \alpha(t)| = r_3$. We claim that unless $r_3 \le \varepsilon/2$, then at least one of the two subarcs of γ_{ab} (before or after t) would have to be too long. The most sensitive case is when the midpoint of α is displaced sideways. But a spherical right triangle of legs r_3 and $\pi/4$ has hypotenuse

$$\arccos\left(\frac{\cos(r_3)}{\sqrt{2}}\right) \le \frac{1}{2}\sqrt{\frac{\pi^2}{4} + 4r_3^2}.$$

This cannot exceed $\delta/2$ or $\hat\gamma$ would be too long. So $r_3 \le \varepsilon/2$.

Summarizing, the C^0 distance from K to the round circle is at most $r_1 + r_2 + r_3 \le 2\varepsilon$. □

Note that we have not tried to get the optimal constant in the theorem above. But the power law cannot be improved: even in the plane there is an example [DGR04] of a curve of distortion $\sqrt{\pi^2/4 + \varepsilon^2}$ whose C^0 distance from the closest round circle is $1 + \frac{3}{2\pi\sqrt{2}}\varepsilon + O(\varepsilon^2)$.

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