1 Introduction

This seminar will be loosely directed at the study of dynamics and moduli spaces, both classical topics and current developments.

Its topics will range from the concrete dynamics of billiards in pentagons to the abstract theory of flows on homogeneous space to parallels with dynamics on moduli spaces of Riemann surfaces and holomorphic 1–forms.

We will start with background on extremal length, Teichmüller’s theorem, and applications to complex dynamics. We will then turn to hyperbolic manifolds in dimensions 2 and 3, and Ratner’s theorem, to set the stage for later developments.

Unique ergodicity. (Cf. [FLP].)

**Theorem 1.1** Let $F$ be an oriented foliation of $X$ with dense leaves. Then the map from the space of transverse invariant measures on $F$ to $H^1(X, \mathbb{R})$ is injective.

**Proof.** Consider a closed transversal $\tau = [a, b]$. Follow the leaf of $F$ through $b$ until it comes back near to $a$ and close it to obtain a cycle $C$. Then the transverse measure $\mu(\tau)$ is nearly the same as the intersection number of $(F, \mu)$ with $C$. Thus $[F, \mu] \in H^1(X, \mathbb{R})$ determines $\mu(\tau)$ for all $\tau$ and hence it determines $\mu$.

**Theorem 1.2** If $F$ is invariant under a pseudo–Anosov mapping $f$, then it is uniquely ergodic.

**Proof.** Let $M \subset H^1(X, \mathbb{R})$ denote the space of cohomology classes represented by transverse invariant measures for $F$. We must show that $M$ reduces to a ray through the standard class $F = [F, \alpha]$. Note that $F$ is an expanding eigenvalue for $f^*$.

Let $F' = [F', \alpha']$ denote the orthogonal measured foliation. Then $M$ lies in the half-space of classes with positive intersection with $P$. Moreover $M$ is a convex cone with a compact base, and it is invariant under $f^*$. Passing to the project space $\mathbb{P}H^1(X, \mathbb{R})$, we obtain a compact convex set $\mathbb{P}M$, invariant under $f^*$, that contains an attracting fixed point $[F]$. This contradicts the
Schwarz lemma for the Hilbert metric on $\mathbb{P}M$, unless the latter space reduces to a point.

**Remarks on $M$.** Let $C(X)$ be the continuous functions on a compact metric space $X$, let $T : X \to X$ be a homeomorphism, and let $P \subset M(X) = C(X)^*$ be the space of $T$–invariant probability measures on $X$. Clearly $P$ is a compact, convex set.

It is known that $P$ is nonempty (since $\mathbb{Z}$ is amenable), and that its extreme points coincide with the ergodic measures. Moreover, $P$ is a Choquet simplex. That is, every point of $P$ can be described uniquely as the barycenter of a measure on its extreme points.

In particular, when $P$ is finite dimensional, it is a simplex.

Returning to the case of an oriented foliation $\mathcal{F}$ as above, the cohomology classes $M \subset H^1(X, \mathbb{R})$ of transverse invariant measures, normalized to have fixed intersection number with the foliation normal to $\mathcal{F}$, is also a simplex.

The span of $M$ is a Lagrangian subspace, so its dimension is at most $g$.

In particular, there are at most $g$ ergodic measures on $\mathcal{F}$. This bound is optimal — it can actually be achieved by variants of the constructions to be given below.

**Failure of unique ergodicity.** Most measured laminations are uniquely ergodic. Indeed, UE has full measure in $\mathbb{P}\mathcal{ML}_g$, and it is known that $\mathcal{F}(e^{2\pi i \theta} q)$ is uniquely ergodic outside an exceptional set of $\theta \in \mathbb{R}/\mathbb{Z}$ with Hausdorff dimension at most $1/2$.

Also, UE is a dense $G_\delta$. Indeed, NUE can be written as a union of closed sets $F_n$ where the failure of UE gives rise to a simplex in $\mathbb{P}\mathcal{ML}_g$ of diameter at least $1/n$; thus NUE is an $F_\sigma$ set. So NUE is rare in both measure and category.

Nevertheless, examples of laminations in NUE can be readily constructed. This construction is in fact easier than the classical construction of NUE interval exchange transformations. Of course a finite union of two or more disjoint simple closed curves is NUE; the real goal is to find a NUE example where every leaf is dense, i.e. a minimal, NUE example.

**Construction with laminations.**

**Theorem 1.3** For every $g \geq 2$ there exists a NUE lamination $\lambda \in \mathbb{P}\mathcal{ML}_g$.

**Proof.** Let $\gamma_1, \gamma_2, \ldots$ enumerate all the simple closed curves on $\Sigma_g$. Since $g \geq 2$, we can choose a pair of disjoint simple closed curves $\alpha_0$ and $\beta_0$ on $\Sigma_g$. Our aim is to construct a sequence of laminations

$$\lambda_n = \alpha_n \cup \beta_n$$
in $\mathcal{PML}_g$, each a union of two simple curves, such that (i) $\alpha_n$ and $\beta_n$ have distinct limits, $\alpha$ and $\beta$; and (ii) $\lambda = \alpha \cup \beta$ meets $\gamma_i$ for every $i$. The last property insures that $\lambda$ is minimal, and by construction it carries two different invariant measures, $\alpha$ and $\beta$.

Notice that the condition $i(\lambda, \gamma_i) \neq 0$ is open. So if insure that $\lambda_i$ meets $\gamma_i$, and inductively choose $\lambda_{i+1}$ very close to $\lambda_i$ — close enough that previous intersections with $\gamma_j$, $j \leq i$ are maintained, and even maintained in the limit $\lambda$ — then $\lambda$ will be minimal.

Let us detail now the inductive construction. If $\lambda_i$ crosses $\gamma_{i+1}$, there is nothing to do – we set $\lambda_{i+1} = \lambda_i$.

Otherwise, $\gamma_{i+1}$ is disjoint from $\alpha_i$ and $\beta_i$. Cut $\Sigma_g$ open along $\beta_i$, choose a complicated simple curve $\delta_i$, and reglue. Then $\delta_i$ meets $\alpha_i$ and $\gamma_{i+1}$ but not $\beta_i$. Let $\alpha_{i+1} = \tau^n(\delta_i)$, where $\tau$ is a Dehn twist around $\alpha_i$. By choosing $n$ large, we can insure that $\alpha_{i+1}$ is as close to $\alpha_i$ as we like. It remains disjoint from $\beta_i$, and continues to meet $\gamma_{i+1}$, so we can now simply set

$$\alpha_{i+1} = \tau^n(\delta_i), \quad n \gg 0,$$

and $\lambda_{i+1} = \alpha_{i+1} \cup \beta_i$.

A special case arises when $\gamma_{i+1}$ coincides with a component of $\lambda_i$, say with $\alpha_i$. But then we proceed exactly as above, with $\delta_i$ crossing $\alpha_i$ but not $\beta_i$.

Since we have moved $\alpha_0$ and $\beta_0$ only by a tiny amount, we can insure that $\alpha \neq \beta$ in the limit. Then $\lambda = \alpha \cup \beta$ is the desired lamination.

Construction with connect sums. Here is a construction in the spirit of Veech’s original examples [V]. (Veech actually worked with an irrational rotation on $S^1 \times \mathbb{Z}/2$ that swaps circles when $x$ lands in a fixed interval $[0, \theta]$.)

Let $E = (\mathbb{C}/\Lambda, dz)$ be a torus equipped with a 1-form. Given an embedded arc $I = [0, z]$ in $E$, we can slit $E$ open along $I$, and glue together two copies of the result to obtain a 1-form of genus two:

$$(X, \omega) = E \oplus_I E.$$

There is a natural map $\pi : X \to E$, branched over $\{0, z\}$, such that $\omega = \pi^*(dz)$. The preimage of $I$ gives a loop $J \subset X$ cutting $X$ into two pieces $A$ and $B$, each a lift of $E - I$. Their areas satisfy $|A| = |B|$.

We will show:

**Theorem 1.4** If $I$ does not lie on a closed geodesic in $E$, then the foliation $\mathcal{F}(e^{i\theta} \omega)$ is minimal, but not uniquely ergodic, for uncountably many values of $\theta$.  

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In fact for the $\theta$ we construct, $X$ can be decomposed into two sets of equal measure, each a union of leaves of $\mathcal{F}(e^{i\theta} \omega)$.

To begin the construction, let $I' = [0, z']$, where $z' = z \mod 2\Lambda$, and assume $I'$ injects into $E$. Note that $I$ and $I'$ have the same endpoints; moreover, the cycle $I + I'$ is trivial in $H^1(E, \mathbb{Z}/2)$, so $I$ and $I'$ determine the same double cover of $X$ and hence the same form $(X, \omega)$.

Of course $A'$ is generally different from $A$. But if the determinant of the parallelogram spanned by $z$ and $z'$ is small, then these two sets are close in the sense of measure.

**Theorem 1.5** We can order $(A', B')$ so that $|A \triangle A'| \leq |\text{Im}(z, z')|$.  

**Proof.** Consider the triangle $T \subset \mathbb{C}$ with vertices $0, z$ and $z' = z + 2\lambda$, $\lambda \in \Lambda$. Its area is given by $d/2$, where $A = |\text{Im}(z, z')|$. Since the edge from $z$ to $z'$ double covers its projection to $E$, we find that the projection of $T$ to $E$ gives a $\mathbb{Z}/2$ cycle $T_E$ with $\partial T_E = I + I'$. Similarly, the preimage $T_X$ of $T_E$ on $X$ satisfies $\partial T_X = J + J'$. The mass of these cycles satisfies

$$|T_X| \leq 2|T_E| \leq 2|T| = d,$$

and $\partial(A + T_X) = I + (I + I') = I'$, so $A + T_X = A'$ and the theorem follows. 

**Lemma 1.6** If $I$ does not lie along a closed geodesic in $E$, then for any $\epsilon > 0$ there exists a $z' \neq z + 2\Lambda$ such that $0 < |\text{Im}(z, z')| < \epsilon$.  

**Proof.** The set $\text{Im}(z + 2\Lambda)$ is dense in $\mathbb{R}$, since it is the image of $\Lambda$ under a group homomorphism with trivial kernel. 

**Proof of Theorem 1.4.** For concreteness let $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$ so that $E$ is the square torus, and choose $z_0 \neq 0$ so its slope $\text{Im}(z_0)/\text{Re}(z_0)$ is irrational. Then by the results above, we can choose $z_i \in z_0 + 2\Lambda$, $i = 1, 2, 3, \ldots$ such that $0 < |\text{Im}(z_i z_{i+1})| < 1/2^i$, such that

$$z_i/z_i \to z_\infty \in S^1,$$

and such that $z_\infty$ is not parallel to any relative period of $(X, \omega)$. Then the foliation $\mathcal{F}_\infty$ of $(X, |\omega|)$ at slope $z_\infty$ is minimal, since it has no saddle connections. Moreover for each $i$ we have a set $A_i \subset X$ of measure $|A_i| = |X|/2$, which is a union of leaves of the foliation $\mathcal{F}_i$ of $X$ parallel to $z_i$. Since
\[ \sum 2^{-i} < \infty, \] the sets form a Cauchy sequence and converge to set \( A \subset X \).

This limiting set is saturated with respect to the limiting folation \( \mathcal{F}_\infty \), and still occupies only \( 1/2 \) the area of \( X \), so \( \mathcal{F}_\infty \) is not ergodic (and therefore not uniquely ergodic).

**Remark.** This argument can be pushed farther to construct orientable foliations in genus \( g \) with \( g \) linearly independent ergodic measures.

## 2 Problems

1. Show that for every \( \delta \) with \( 0 < \delta < 1 \), there exists an \( x \in S^1 = \mathbb{R}/\mathbb{Z} \) such that \( E_x = \frac{2\pi x}{\delta} \subset S^1 \) is a Cantor set of Hausdorff dimension \( \delta \).

2. Show that every compact hyperbolic surface can be constructed by gluing together the sides of a convex hyperbolic polygon with isometries.

3. Let \( (X, \omega) \) be a nonzero holomorphic 1–form with \( X \) a compact Riemann surface (of genus \( g \geq 1 \)). Can \( (X, \omega) \) always be constructed by gluing together the sides of a convex polygon \( (P, dz) \) in the plane by translation? If not, can we at least choose \( P \) to be connected?

4. Prove that the center of \( \text{Mod}_g \) is generated by the hyperelliptic involution when \( g = 1, 2 \) and that it is trivial for \( g \geq 3 \).

5. Prove that there are only finitely many types of simple closed curves on a surface of genus \( g \) — up to the action of the mapping–class group.

6. *Use the preceding fact to prove that \( \text{Mod}_g \) is generated by Dehn twists.

7. *Use the fact that \( \text{Mod}_g \) is generated by Dehn twists to prove that every compact orientable 3–manifold \( M \) is obtained by Dehn surgery on a link in \( S^3 \). (Hint: first prove that \( M \) has a Heegaard splitting.)

8. Formulate a conjecture of the form: \( s \) is the slope of a periodic trajectory in the regular pentagon (with one edge vertical) if and only if \( x \in E \), where \( E \) has something to do with \( \mathbb{Q}(\sqrt{5}) \). Prove one or *both directions of your conjecture.

9. **Find the periodic slopes in the regular 7-gon.
10. Let \( \Gamma_n \subset \text{SL}_2(\mathbb{R}) \) be the (Hecke) subgroup generated by \( \begin{pmatrix} 1 & 2 \cos(\pi/n) \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

(a) Describe \( \mathbb{H}/\Gamma_n \) as an orbifold.
(b) Show that \( \Gamma_n \) is discrete for all \( n \geq 3 \).
(c) Show that \( \Gamma_3 = \text{SL}_2(\mathbb{Z}) \).
(d) * Show that \( \Gamma_5 \cdot \infty = \mathbb{Q}(\sqrt{5}) \cup \{\infty\} \).
(e) ** Determine the cusps of \( \Gamma_7 \).

11. Define a natural symplectic form on the variety of (discrete, faithful) representations of \( \pi_1(\Sigma_g) \) into \( \text{SL}_2(\mathbb{R}) \) (up to conjugacy). (Hint: what is the tangent space?)

12. Given \( \lambda > 1 \): use the method of extremal length to estimate above and below, with concrete constants, the value of \( t > 0 \) such that the upper half plane, with the points 0, 1, \( \lambda \) and \( \infty \) marked, is conformally equivalent to a rectangle of the form \( R = \{x+iy : x \in [0,1], y \in [0,t]\} \), with \([0,1]\) corresponding to \([0,1]\).

13. Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map such that \( f \) sends \( \{0, 1, \infty, a\} \) into \( \{0,1,\infty\} \). Show that \( a \) is an algebraic number.

14. Let \( E = \mathbb{C}/\mathbb{Z} \oplus \tau \mathbb{Z} \), and let \( \Gamma \) denote the set of loops on \( E \) with period \( a + b\tau \). Compute the extremal length \( \lambda(E, \Gamma) \) and show its level sets are parallel horocycles. Explain the significance of the point on \( \partial \mathbb{H} \) where these horocycles rest.

15. Prove that the map \( f : \mathbb{C} \to \mathbb{C} \) given in polar coordinates by \( f(r, \theta) = (r^\alpha, \theta) \) with \( 0 < \alpha < 1 \) is \( K \)-quasiconformal, where \( K = 1/\alpha \). Show that there exists a circle \( C \) defined by \(|z - c| = r\) such that the maximum and minimum distances of \( f(C) \) from \( f(c) \) satisfy \( M/m > K \).

16. Prove that a \( K \)-quasiconformal map is \( 1/K \)-Hölder continuous.

17. Construct explicitly, in genus \( g \), a family of Strebel differentials of dimension \( 6g - 5 = 1 + \dim \mathcal{M}_g \).

18. Prove that for any finite set \( A \subset \mathbb{C}, |A| \geq 2 \), there exists a rational map \( f \) such that the critical values of \( f \) coincide with \( A \).
19. For each $n \geq 2$, find an explicit polynomial $f_n(z)$ with postcritical set
$P(f_n) = \{0, 1, n, \infty\}$. Prove that for any solution to this problem, we
have $\deg(f_n) \to \infty$. *Give an explicit lower bound on $\deg(f_n)$.

20. Using the classical Jacobi elliptic functions $sn$, $cn$ and $dn$, give an
explicit formula for $f(x)$, $x \in [-1, 1]$, where $f$ is the extremal quasi-
conformal map on $\hat{\mathbb{C}}$ that preserves the real axis, fixes $\pm 1$, and sends
$\pm k$ to $\pm k'$, with $k, k' > 0$.

21. Let $T : X \to X$ be a homeomorphism on a compact metric space
$X$, and let $P \subset C(X)^* = M(X)$ denote the space of $T$–invariant
probability measures.

Show that $P$ is a nonempty, compact convex set, and that its extreme
points $E$ correspond to ergodic measures. Show that the elements of
$E$ are linearly independent.

*Show that $P$ is a Choquet simplex: every point of $P$ is the barycenter
of a unique probability measure on $E$.

References


[V] W. A. Veech. Strict ergodicity in zero dimensional dynamical systems
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