Billiards, quadrilaterals and moduli spaces

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Contents

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1 Introduction

This paper investigates the interaction between Teichmüller theory and algebraic geometry mediated by Hodge theory and periods. Its main goal is to present several remarkable, newly discovered subvarieties of low–dimensional moduli spaces via a unified approach based on Euclidean quadrilaterals.

Totally geodesics subvarieties. Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces X of genus g. If we also record n unordered marked points on X, we obtain the moduli space $\mathcal{M}_{g,n}$.

A subvariety V of moduli space is *totally geodesic* if it contains every Teichmüller geodesic that is tangent to it. It is *primitive* if it does not arise from a lower–dimensional moduli space via a covering construction.

The first family of primitive, totally geodesic varieties of dimension one in $\mathcal{M}_g$ was discovered by Veech in the 1980s [V2]. These rare and remarkable *Teichmüller curves* are related to Jacobians with real multiplication and polygonal billiard tables with optimal dynamical properties. A second family was discovered shortly thereafter [Wa]. To date only a handful of families of Teichmüller curves are known.

The first known primitive, totally geodesic variety of dimension bigger than one is the recently discovered *flex surface* $F \subset \mathcal{M}_{1,3}$ [MMW]. The surface F is closely related to a new type of $\mathrm{SL}_2(\mathbb{R})$–invariant subvariety ΩG in the moduli space of holomorphic 1–forms ΩM_4, called the *gothic locus*. These examples arise from an unexpected interplay between Teichmüller theory and the algebraic geometry of cubic curves in the plane and space curves of genus four.

The Veech and Ward examples are naturally associated to triangles with inner angles proportional to $(1, 1, n)$ and $(1, 2, n)$, with $n \geq 1$. In this paper we show the flex surface and the gothic locus belong to a suite of examples that are naturally associated, in a similar way, to six types of quadrilaterals. This suite yields:

1. Six examples of $\mathrm{SL}_2(\mathbb{R})$–invariant 4-folds in $\Omega\mathcal{M}_g$, for various g;

2. Three examples of primitive, totally geodesic surfaces in $\mathcal{M}_{g,n}$, for various (g, n);

3. Two distinct series of Teichmüller curves in $\mathcal{M}_4$; and

4. Two families of quadrilateral billiard tables with optimal dynamical properties.

Our goal is to provide a unified treatment of these examples, and explain their connection to dihedral curves $X \to \mathbb{P}^1$.

(a_1, a_2, a_3, a_4)	(m, n)	Stratum for ΩG	$F_{m,n} \subset \mathcal{M}_{g',n'}$
$(1, 1, 1, 7)$	$(5, 3)$	$\Omega\mathcal{M}_4(6)$	
$(1, 1, 1, 9)$	$(6, 3)$	$\Omega\mathcal{M}_4(2^3)$	$\mathcal{M}_{1,3}$
$(1, 1, 2, 8)$	$(6, 4)$	$\Omega\mathcal{M}_4(3^2)$	
$(1, 1, 2, 12)$	$(8, 4)$	$\Omega\mathcal{M}_5(2^4)$	$\mathcal{M}_{1,4}$
$(1, 2, 2, 11)$	$(8, 5)$	$\Omega\mathcal{M}_6(10)$	
$(1, 2, 2, 15)$	$(10, 5)$	$\Omega\mathcal{M}_6(2^5)$	$\mathcal{M}_{2,1}$

Table 1. Six types of quadrilaterals and their associated varieties.

Six types of quadrilaterals. It is convenient to index our six examples by quadruples of integers $a = (a_1, a_2, a_3, a_4)$, each of which describes a family of quadrilaterals and two families of holomorphic 1–forms: the cyclic forms, and their saturation under the action of $\mathrm{SL}_2(\mathbb{R})$. We require that the integers (a_1, a_2, a_3, a_4) are positive and relatively prime, that $\sum a_i = 2m$ is even, and that $a_i \neq m$ for all i.

The quadrilaterals of type (a_1, a_2, a_3, a_4) are those with internal angles $\pi a_i/m$. The angles can appear in any order around the quadrilateral.

Cyclic forms. A *cyclic form* (X, ω) of type (a_1, a_2, a_3, a_4) is specified by a quadruple of distinct points $(b_1, \ldots, b_4)$ in $\mathbb{C}$. This data determines a compact Riemann surface X, defined by the equation

$$y^m = \prod_1^4 (x - b_i)^{m-a_i}, \tag{1.1}$$

and the corresponding holomorphic 1–form is simply $\omega = dx/y$. Equation (1.1) presents X as a cyclic branched covering of $\mathbb{P}^1$, and ω is an eigenform for the deck group $\mathbb{Z}/m \subset \mathrm{Aut}(X)$ generated by $r(x, y) = (x, \zeta_m y)$; in fact

$$r^*\omega = \zeta_m^{-1}\omega,$$

where $\zeta_m = \exp(2\pi i/m)$.

The set of all forms of type $a = (a_1, a_2, a_3, a_4)$ is a 2–dimensional algebraic subvariety $\Omega Z_a \subset \Omega\mathcal{M}_g$, where

$$2g-2 = \sum_1^4 a_i - \gcd(a_i, m).$$

Note that $|dx/y|$ defines a flat metric on $\mathbb{P}^1$ with a conical singularity of angle $2\pi a_i/m$ at each branch point b_i. In particular, when the branch points are real, $(\mathbb{P}^1, |dx/y|)$ is isometric to the double of a quadrilateral of type a.

We will focus on quadrilaterals where

$$1 = a_1 \le a_2 \le a_3 \le 2.$$

Such a quadrilateral type is uniquely determined by the pair of integers

$$(m,n) = \left(\frac{1}{2}\sum a_i,\ a_1+a_2+a_3\right). \tag{1.2}$$

We also require that $(a_1, a_2, a_3) = (1,1,1)$ if m is odd. These conditions insure that the divisor of a cyclic form ω is a multiple of the *special fiber* $X^* \subset X$ over b_4.

Invariant subvarieties. We define the *saturation* of the space of cyclic forms by:

$$\Omega G_a = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot \Omega Z_a} \subset \Omega\mathcal{M}_g.$$

This is the smallest closed, invariant set of forms containing the cyclic ones.

Our first result describes this space and its properties.

Theorem 1.1 *For each of the six values of $a = (a_1, a_2, a_3, a_4)$ listed in Table 1, the saturation of the cyclic forms gives a primitive, irreducible, 4–dimensional invariant subvariety $\Omega G_a \subset \Omega\mathcal{M}_g$.*

To be more explicit, in §3 we will define the variety of *dihedral forms* $\Omega D_{m,n}$, and in §5 we will show:

Theorem 1.2 *In all six cases, the dihedral forms of type (m,n) form a Zariski open subset of ΩG_a.*

Here (m,n) is related to (a_1, a_2, a_3, a_4) by equation (1.2) (see Table 1). Thus one could alternatively define this new invariant locus by:

$$\Omega G_a = \overline{\Omega D}_{m,n}. \tag{1.3}$$

Algebraic correspondences. The remarkable feature of the six cases listed in Table 1 is that, while the action of $\mathrm{SL}_2(\mathbb{R})$ destroys the cyclic symmetry of forms in ΩZ_a, it merely deforms the correspondence $T(x) = r^{\pm 1}x$ as an algebraic cycle in $X \times X$. The relation

$$T^*\omega = (\zeta_m + \zeta_m^{-1})\omega$$

persists under deformation, and gives rise to a natural action of the dihedral group D_{2m} on a degree two extension $Y \to X$.

Linearity in period coordinates. The main step in the proof of these theorems is to show that $\Omega D_{m,n}$ is locally defined by real linear equations in period coordinates (§5).

This is accomplished by combining an upper bound on the linear span of the space of dihedral forms (§3) with a lower bound on the dimension of $\Omega D_{m,n}$ as an algebraic variety (§4). Along the way we obtain explicit formulas for dihedral forms. We also use the rigidity theorems of [EMM] to show that ΩG_a is an analytic variety; this could be avoided if we took equation (1.3) as its definition.

Totally geodesic curves and surfaces. Next we examine varieties that arise only for special values of (m, n), and state two results proved in §6.

When $m = 2n$, the locus $\Omega D_{m,n}$ maps into a closed, 2–dimensional subvariety $F_{m,n}$ of a moduli space of lower dimension, indicated in the last column of Table 1. This map is constructed by sending each dihedral form to a quadratic differential $(A, q) = (X, \omega^2)/r^{m/2}$, and then recording the quotient surface A and the poles of q.

Theorem 1.3 *The subvarieties*

$$F_{6,3} \subset \mathcal{M}_{1,3}, \quad F_{8,4} \subset \mathcal{M}_{1,4} \quad \textit{and} \quad F_{10,5} \subset \mathcal{M}_{2,1}$$

are primitive, irreducible, totally geodesic surfaces in their respective moduli spaces.

When $m = 6$, the dense subvariety $\Omega D_{m,n} \subset \Omega G_a$ is locally defined over $\mathbb{Q}$ in period coordinates, and we can define finer $\mathrm{SL}_2(\mathbb{R})$–invariant loci inside $\Omega D_{m,n}$ by imposing the condition that (X, ω) is an eigenform for real multiplication. As a result we obtain (§7):

Theorem 1.4 *The 3-folds G_{1119} and G_{1128} in $\mathcal{M}_4$ each contain a dense set of primitive Teichmüller curves.*

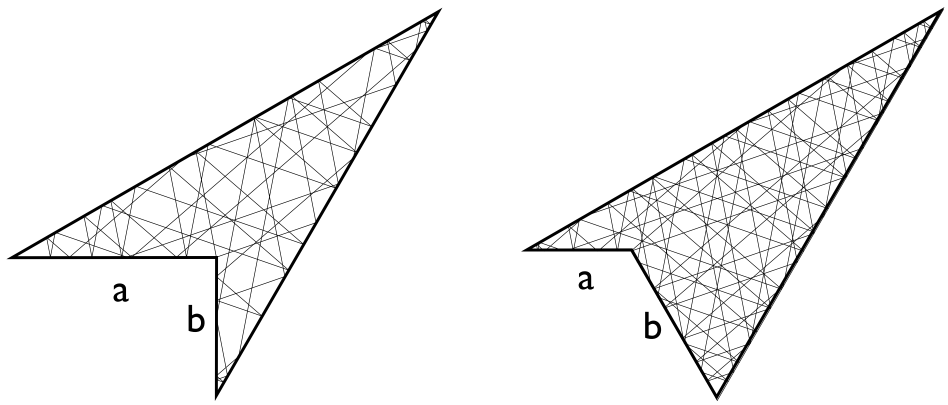

Figure 2. The quadrilaterals $Q_{1119}(a, b)$ and $Q_{1128}(a, b)$. A closed billiard trajectory is shown in each.

Here G_a denotes the projection of ΩG_a to $\mathcal{M}_g$.

Billiards. Finally we consider the quadrilaterals of type $(1, 1, 1, 9)$ and $(1, 1, 2, 8)$, shown in Figure 2. We say billiards in a polygon has *optimal dynamics* if every trajectory is either periodic or uniformly distributed (cf. [V2], [D]). In §8 we will show:

Theorem 1.5 *Billiards in the quadrilateral $Q_{1119}(1, \sqrt{3}y)$ has optimal dynamics, and the associated cyclic 1–form generates a primitive Teichmüller curve in $\mathcal{M}_4$, provided $y > 0$ is irrational and*

$$y^2 + (3c + 1)y + c = 0$$

for some $c \in \mathbb{Q}$. The same is true for $Q_{1128}(1, y)$, provided $y^2+(2c+1)y+c = 0$.

These quadrilaterals give explicit Teichmüller curves in G_{1119} and G_{1128}. The cases $c = -1/4$ and $c = -1$ are shown at the left and right, respectively, in Figure 2.

In both cases, if y is *rational*, then the corresponding quadrilateral still has optimal dynamics, but the associated cyclic 1-form has periods in $\mathbb{Q}(\sqrt{-3})$; it is pulled back from the hexagonal torus, and the corresponding Teichmüller curve is not primitive.

The conditions on y above come from the fact that the two Galois conjugate eigenforms for real multiplication must be orthogonal, with respect to the Hermitian form on periods giving the area of the quadrilateral Q. See §8.

Relation to known examples. As mentioned above, the suite of examples just described includes the flex locus and the gothic locus of [MMW]; these are given by $F_{6,3}$ and $\Omega G_{6,3}$ respectively. It also presents, from a different perspective, the family of Teichmüller curves lying in G_{1119} described above. The isosceles billiard table $Q_{1119}(1,1)$ is essential the same as the Ward triangle with angles proportional to $(1,2,9)$.

All the remaining examples of $\mathrm{SL}_2(\mathbb{R})$–invariant subvarieties, totally geodesic surfaces, primitive Teichmüller curves and optimal billiard tables described above are new.

Additional examples of quadrilateral billiard tables with optimal dynamics are given in [BM, §7]. That paper exhibits a single optimal quadrilateral table of type $a = (a_1, a_2, a_3, a_4)$, for each type of the form

$$\begin{aligned} a &= (2,2,n,3n-4), \quad \text{with } n \geq 5 \text{ and } n \text{ odd; and} \\ a &= (1,2,2,4n-5), \quad \text{with } n \geq 7, \text{ and } n = 1,3,7 \text{ or } 9 \bmod 5. \end{aligned}$$

Theorem 1.5, on the other hand, gives infinitely many optimal quadrilateral tables of 2 fixed types, namely $(1,1,1,9)$ and $(1,1,2,8)$.

The dihedral approach also gives a new perspective on the Teichmüller curves in $\mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_4$ constructed using real multiplication and Prym varieties in [Mc1] and [Mc3]; see §7.

Listing quadrilaterals. A conceptual approach to the list of quadrilaterals in Table 1 will be described in §4. In brief we consider, for $n = 3, 4$ and 5, all m such that $n + \gcd(m,n) \leq m \leq 2n$, and then eliminate from this list those $\Omega G_{m,n}$ that arise from strata. The crucial bound $m \leq 2n$ insures that $\dim \Omega D_{m,n} = 4$.

Triangles and pentagons. Our methods can be applied to polygons with any number of sides. In the case of triangles, we recover the Veech and Ward examples with $(a_1, a_2, a_3) = (1, 1, m-2)$ and $(1, 2, m-3)$; see Appendix A. Equation (A.6) gives the first explicit algebraic formula for the Ward examples. The $(3,5,7)$ lattice triangle gives a Teichmüller curve in $G_{1,1,1,7}$; see §5. A complete list of lattice triangles is still unknown; see [Ho] for a summary of current knowledge.

In the case of pentagons, our methods lead to one new example, described in Appendix B.

Theorem 1.6 *The cyclic forms of type $a = (1,1,2,2,12)$ generate a primitive, 6–dimensional $\mathrm{SL}_2(\mathbb{R})$ invariant subvariety of $\Omega\mathcal{M}_4$.*

We emphasize that we have only studied a restricted class of polygons, so our surveys are far from complete, even for triangles.

Notes and references. Mirzakhani conjectured that the only invariant subvarieties of $\Omega\mathcal{M}_g$ of rank two or more, in the sense of [Wr2], are those coming from strata of 1–forms or quadratic differentials [ANW, Conj. 1.3].

The six examples ΩG_a listed in Table 1 are the first known counterexamples to this conjecture (see §5). It would be interesting to find more varieties of this type (Theorem 1.6 gives one more).

We would like to acknowledge helpful conversations with Mirzakhani in the early stages of this project, which contributed to the discovery of the first example in Table 1.

Notation. We let $H^n(A)$ and $H^n(A,B)$ denote absolute and relative cohomology with complex coefficients; when coefficients are in K, we write $H^n(A;K)$ and $H^n(A,B;K)$. We use exponential notation for repeated indicates; e.g. $\Omega\mathcal{M}_4(2^3) = \Omega\mathcal{M}_4(2,2,2)$ in Table 1.

2 Cyclic forms

In this section we define the variety of cyclic forms ΩZ_a associated to a general polygon type $a = (a_1, \ldots, a_N)$, compute its dimension and prove it is locally linear in period coordinates. The latter argument is a model for the proof of linearity of $\Omega D_{m,n}$, which will be carried out in §§3, 4 and 5.

Recall that a variety V is *unirational* if there exists a dominant rational map $u : \mathbb{P}^d \to V$ for some d. The main result of this section is:

Theorem 2.1 *The locus $\Omega Z_a \subset \Omega\mathcal{M}_g$ is a unirational variety of dimension $N-2$, locally defined by complex linear equations in period coordinates.*

The defining equations for ΩZ_a are given in Theorem 2.5 below. In the course of the proof we will also show that the projection of ΩZ_a to absolute periods $H^1(X)$ has fibers of dimension

$$\epsilon(a) = |\{1 \le i \le N \ : \ m|a_i\}|. \tag{2.1}$$

See Theorem 2.3. For example, when $N = 4$ the projection always is locally injective.

Strata and period coordinates. We begin by recalling some background material. Let $\Omega\mathcal{M}_g$ denote the moduli space of nonzero holomorphic 1–forms (X,ω) with $X \in \mathcal{M}_g$. The zero set of a form will be denoted by $Z(\omega) \subset X$.

The space $\Omega\mathcal{M}_g$ breaks up into *strata* $\Omega\mathcal{M}_g(p_1,\ldots,p_n)$, indexed by partitions of $2g-2$, consisting of those forms (X,ω) whose n zeros have multiplicities $p_1,\ldots,p_n$. The bundle of topological pairs $(X,Z(\omega))$ is locally trivial over each stratum. Thus on a small (orbifold) neighborhood U of (X,ω) in its stratum, we can define *period coordinates*

$$p : U \to H^1(X, Z(\omega))$$

sending $(X',\omega') \in U$ to the cohomology class $[\omega']$. These maps are holomorphic local homeomorphisms (see e.g. [Y, §6]).

Next we proceed to the definition of the space of cyclic forms ΩZ_a. The general case is consistent with definition given by equation (1.1) in the Introduction.

Polygon types. A *polygon type* is a sequence of positive integers $a = (a_1,\ldots,a_N)$, such that

$$\sum_1^N a_i = (N-2)m \tag{2.2}$$

for some integer $m > 0$, and $\gcd(a_1,\ldots,a_N,m) = 1$. We also require $a_i \neq m$ for all i. One can think of a as specifying the shape of a Euclidean N-gon with internal angles $\theta_i = \pi a_i/m$. We allow $\theta_i \geq 2\pi$ but we do not allow $\theta_i = \pi$.

Cyclic triples and cyclic forms. Fix a polygon type a, and let $\zeta_m = \exp(2\pi i/m)$. A *cyclic triple* (X,ω,r) of type a is a holomorphic 1-form $(X,\omega) \in \Omega\mathcal{M}_g$, together with an automorphism $r \in \mathrm{Aut}(X)$ of order m, such that:

1. X/r has genus 0;
2. We have $r^*\omega = \zeta_m^{-1}\omega$; and
3. The divisor of the pluricanonical form $(\mathbb{P}^1,\xi) = (X,\omega^m)/r$ can be written as
$$(\xi) = \sum_1^N (a_i - m) b_i, \tag{2.3}$$
where $b_1,\ldots,b_N$ are distinct points in $\mathbb{P}^1$.

In this case (X,ω) itself is a *cyclic form* of type a. The set of all such forms will be denoted by $\Omega Z_a \subset \Omega\mathcal{M}_g$.

Equation (2.3) implies that $|\omega| = |\xi|^{1/m}$ gives a metric on $\mathbb{P}^1$ with cone angles $2\pi a_i/m$ at the points b_i, in the spirit of [Th].

The zeros of a cyclic form. Let $X_i \subset X$ denote the fiber over b_i of the natural projection $\pi : X \to X/r \cong \mathbb{P}^1$. It is easy to check that $|X_i| = f_i = \gcd(a_i, m)$, and that the divisor of a cyclic form is given by

$$(\omega) = \sum e_i X_i$$

where $e_i f_i = a_i - \gcd(a_i, m)$. In particular, the genus g of X satisfies

$$2g - 2 = \sum_1^N a_i - \gcd(a_i, m), \tag{2.4}$$

and ΩZ_a is contained in a single stratum of $\Omega\mathcal{M}_g$. Note that ω has no zeros on X_i if a_i divides m.

The variety of cyclic triples. It is useful to introduce the space $\widetilde{\Omega} Z_a$ of all cyclic triples (X, ω, r) of type a. Since $r \in \mathrm{Aut}(X)$ and the latter group is finite, the forgetful map $\widetilde{\Omega} Z_a \to \Omega Z_a$ is finite–to–one. There is a natural action of $\mathbb{C}^*$ on both spaces, by rescaling ω, and we denote their quotients by $\mathbb{P}\widetilde{\Omega} Z_a$ and $\mathbb{P}\Omega Z_a$ respectively.

Let $b_{N-1} = 0$, let $b_N = \infty$, and let $U \subset \mathbb{C}^{N-2}$ be the open set of nonzero sequences $b = (b_1, \ldots, b_{N-2})$ such that $b_i \neq b_j$ for $1 \leq i < j \leq N-2$. Let

$$p : U \to \widetilde{\Omega} Z_a$$

be the map that sends b to the cyclic triple (X, ω, r), where X is defined by

$$y^m = \prod_1^{N-1} (x - b_i)^{m - a_i},$$

and where

$$\omega = \frac{dx}{y} \quad \text{and} \quad r(x, y) = (x, \zeta_m y).$$

Note that ω is holomorphic, because $\int_X |\omega|^2 = m \int_{\mathbb{P}^1} |dx/y|^2 < \infty$. We can now show:

Theorem 2.2 *The locus ΩZ_a is a unirational variety of dimension $N - 2$.*

Proof. Let $\widetilde{\mathcal{M}}_{0,N}$ and $\mathcal{M}_{0,N}$ denote the moduli spaces of N ordered and unordered points on $\mathbb{P}^1$ respectively. We then have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "u"'] & \widetilde{\Omega} Z_a \arrow[d] & \\ \widetilde{\mathcal{M}}_{0,N} \arrow[r, "\overline{p}"] & \mathbb{P}\widetilde{\Omega} Z_a \arrow[r, "q"] & \mathcal{M}_{0,N}, \end{tikzcd}$$

where $u(b) = [b_1, \dots, b_{N-2}, 0, \infty]$ and where $q(X, \omega, r)$ is the support of the divisor of the pluricanonical form $(\mathbb{P}^1, \xi) = (X, \omega^m)/r$. Up to a change of coordinates on $\mathbb{P}^1$, the divisor of ξ can always be written in the form (2.3) with $b_{N-1} = 0$ and $b_N = \infty$, so the map $\overline{p}$ is surjective. On the other hand, the composition $q \circ \overline{p} : \widetilde{\mathcal{M}}_{0,N} \to \mathcal{M}_{0,N}$ just forgets the ordering of the points, so it is finite. It follows that $\overline{p}$ is finite. By similar reasoning, using the fact that $a_i \neq m$ for all i, the map $p : U \to \widetilde{\Omega} Z_a$ is surjective and finite. Since U is birational to $\mathbb{P}^{N-2}$, and the map $\widetilde{\Omega} Z_a \to \Omega Z_a$ is surjective and finite, this shows that ΩZ_a is a unirational variety of dimension $N - 2$. ■

Eigenspaces. Given a cyclic triple (X, ω, r), let

$$E(X, \omega, r) = \operatorname{Ker}(r^* - \zeta_m^{-1} I) \subset H^1(X, Z(\omega)) \tag{2.5}$$

denote the eigenspace for r containing $[\omega]$. Next we show that this eigenspace and the algebraic variety ΩZ_a have the same dimension.

Theorem 2.3 *For any cyclic triple of type $(a_1, \dots, a_N)$, we have:*

$$\dim E(X, \omega, r) = N - 2.$$

The projection of $E(X, \omega, r)$ to $H^1(X)$ has fibers of dimension $\epsilon(a)$.

Proof. This type of character calculation is well–known; for an early variant, see [ChW].

The 1–dimensional characters of $\mathbb{Z}/m = \langle r \rangle$ are given by $\chi_j(r) = \zeta_m^j$, $j \in \mathbb{Z}/m$. We say χ_j is *primitive* if $\gcd(m, j) = 1$.

Given $h \in \mathbb{Z}/m$, let $\chi(h) = \operatorname{Tr}(h | H^1(X))$ and let $\operatorname{Fix}(h)$ denote the set of fixed–points of $h \in \mathbb{Z}/m$ acting on X. Recall that these fixed points occur only on the fibers X_i of $\pi : X \to \mathbb{P}^1$ over b_i. Let $\operatorname{Fix}_i(h) = |\operatorname{Fix}(h|X_i)|$ and let $f_i = |X_i| = \operatorname{Fix}_i(r^0) = \gcd(a_i, m)$.

By the Lefschetz trace formula, we have

$$\chi(r^k) = 2 - |\operatorname{Fix}(r^k)| = 2 - \sum_{i=1}^{N} \operatorname{Fix}_i(r^k),$$

except for $k = 0$ we have, using equation (2.4),

$$\chi(r^0) = 2g = 2 + \sum_1^N (a_i - f_i) = 2 + (N-2)m - \sum_1^N \operatorname{Fix}_i(r^0).$$

Since the representation of $\mathbb{Z}/m$ is defined over $\mathbb{Q}$, all primitive representations occur in $H^1(X)$ with the same multiplicity

$$\mu = \langle \chi_1, \chi \rangle = \frac{1}{|\mathbb{Z}/m|} \sum_{\mathbb{Z}/m} \chi_1(h)\chi(h).$$

Since the function $\mathrm{Fix}_i(h)$ on $\mathbb{Z}/m$ has period f_i, we have $\langle \chi_1, \mathrm{Fix}_i \rangle = 0$ unless $f_i = m$, in which case $\langle \chi_1, \mathrm{Fix}_i \rangle = 1$; and of course $\langle \chi_1, 2 \rangle = 0$. The number of i such that $f_i = m$ is given by $\epsilon(a)$, and hence

$$\mu = \langle \chi_1, \chi \rangle = (N-2) - \epsilon(a).$$

On the other hand, using the exact sequence of $\mathbb{Z}/m$–modules

$$\mathbb{C} \to H^0(Z(\omega)) \to H^1(X, Z(\omega)) \to H^1(X) \to 0,$$

and the fact that $H^0(Z(\omega)) \cong \bigoplus_{a_i > f_i} \mathbb{C}[\mathbb{Z}/f_i]$, we conclude that each primitive representation of $\mathbb{Z}/m$ occurs with multiplicity $\epsilon(a)$ in $H^0(Z(\omega))$, and hence with multiplicity $N-2$ in $H^1(X, Z(\omega))$. In particular, $\dim E(X, \omega, r) = N-2$, and the projection of $E(X, \omega, r)$ to $H^1(X)$ has fibers of dimension $\epsilon(a)$. ■

Linearity in period coordinates. Let $V \subset \Omega\mathcal{M}_g$ be an irreducible subvariety of a stratum. Suppose for each form in V we have a subspace in relative cohomology, defined over a number field, such that

$$[\omega] \in S(X, \omega) \subset H^1(X, Z(\omega)).$$

By a variant of the proof of [MMW, Theorem 5.1], we have:

Theorem 2.4 *If $d = \dim(V) \geq \dim S(X, \omega)$ at all points of V, then V is locally linear in period coordinates. More precisely, V locally coincides with a finite union of d–dimensional subspaces of the form*

$$\bigcup_1^s S(X_i, Z(\omega_i)) \subset H^1(X, Z(\omega)).$$

The same is true for the closure of V in its stratum.

(Here the forms (X_i, ω_i) and (X, ω) are in a common period chart.)

As a special case, we can now prove:

Theorem 2.5 *The locus ΩZ_a locally coincides, in period coordinates, with a finite union of subspaces of the form*

$$\bigcup_1^s E(X_i, \omega_i, r_i) \subset H^1(X, Z(\omega)).$$

(The union is over a finite collection of cyclic triples of type a.)

Proof. Let $V = \Omega Z_a$. For each form in ΩZ_a, let $S(X,\omega) = E(X,\omega,r)$ for some r such that (X,ω,r) is a cyclic triple of type a. Then $\dim(V) = \dim S(X,\omega) = N-2$ by Theorems 2.2 and 2.3, so we can apply the preceding result. ∎

Proof of Theorem 2.1. Combine Theorems 2.2 and 2.5. ∎

3 Dihedral forms

We now turn to the study of the space of dihedral forms

$$\Omega D_{m,n} \subset \Omega \mathcal{M}_g.$$

This space is defined for any pair of integers (m,n) with $m \geq 3$ and $0 < n < 6$, subject to the condition that $n = 3$ when m is odd.

Our main goal, achieved in §5, is to show that $\Omega D_{m,n}$ is 4–dimensional and locally linear in period coordinates, for suitable values of m and n.

In this section we will also define the notion of a *dihedral triple* (X,ω,T), where

$$T \subset X \times X$$

is an algebraic correspondence compatible with ω. Every dihedral form (X,ω) comes from a dihedral triple, and T determines an action of D_{2m} on a degree two extension $Y \to X$. The integer n describes the topological type of this action.

We will see that T induces an operator

$$T^* : H^1(X, Z(\omega)) \to H^1(X, Z(\omega)),$$

defined over $\mathbb{Z}$, and that $[\omega]$ belongs to the eigenspace

$$D(X,\omega,T) = \mathrm{Ker}(T^* - \tau_m I), \quad \text{where } \tau_m = \zeta_m + \zeta_m^{-1}. \tag{3.1}$$

The main result of this section is:

Theorem 3.1 *Let (X, ω, T) be a dihedral triple of type (m, n) with $m \neq n$. Then*
$$\dim D(X, \omega, T) = 4,$$
and the projection of $D(X, \omega, T)$ to $H^1(X)$ is injective.

Corollary 3.2 *We have* $\dim \Omega D_{m,n} \leq 4$.

In §5 we will show that when equality holds, the variety $\Omega D_{m,n}$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

Remarks. One can think of T as a deformation of the correspondence
$$x \mapsto r^{\pm 1}(x)$$
associated to a cyclic triple (X, ω, r). It determines a self–adjoint endomorphism of the Jacobian of X with ω as an eigenform. In particular, along the dihedral locus $\mathrm{Jac}(X)$ belongs to a proper Shimura subvariety of $\mathcal{A}_g$, generalizing the Hilbert modular surfaces that appear in [Mc1] and [Mc3].

When $m = n$, X has genus $g = 1$ and $\Omega D_{m,n} = \Omega\mathcal{M}_1$ is two–dimensional.

Dihedral groups. We begin with some definitions leading up to the general idea of a dihedral triple (X, ω, T). We then impose additional conditions to define the locus of dihedral forms $\Omega D_{m,n}$.

Fix $m \geq 1$. The dihedral group of order $2m$ is defined by:
$$D_{2m} = \langle r, f \ : \ r^m = f^2 = (rf)^2 = \mathrm{id} \rangle.$$
Let $\mathbb{Z}/m \subset D_{2m}$ denote the normal subgroup generated by r. Then D_{2m} acts on $\mathbb{Z}/m$ by
$$r \cdot x = x + 1 \quad \text{and} \quad f \cdot x = -x,$$
yielding inclusions
$$D_{2m} \subset S_m \cong \mathrm{Sym}(\mathbb{Z}/m). \tag{3.2}$$

We refer to elements of D_{2m} of the form r^i as *rotations*, and to those of the form fr^i as *reflections*. The conjugacy class of a rotation is given by $[r^i] = \{r^i, r^{-i}\}$. When m is odd, the reflections form a single conjugacy class in D_{2m}; while when m is even, there are two such conjugacy classes, $[f]$ and $[rf]$.

We emphasize that when m is even, the symmetry between f and rf in D_{2m} is broken by the map to S_m; the permutation associated to f has two fixed points, while that associated to rf has none.

Dihedral correspondences. Let X be a compact Riemann surface, and let $T \subset X \times X$ be an irreducible algebraic curve. Assume that T that determines a symmetric correspondence of degree $(2,2)$. Let $Y \to T$ be the normalization of the (generally singular) curve T, and let

$$\xi : Y \to X$$

be the composition of the normalizing map with projection to the first factor of $X \times X$. Since $\deg(\xi) = 2$, there is a unique involution $s_1 \in \mathrm{Aut}(Y)$ satisfying $\xi \circ s_1 = \xi$. Since T is symmetric, there is also a unique involution $s_2 \in \mathrm{Aut}(Y)$ obtained by interchanging the two factors of $X \times X$.

We say (X,T) is a *dihedral correspondence* of type m if the product $s_1 s_2$ has order m in $\mathrm{Aut}(Y)$.

This condition implies we have a canonical inclusion

$$D_{2m} \hookrightarrow \mathrm{Aut}(Y) \tag{3.3}$$

sending f to s_1 and fr to s_2. The correspondence T is simply the image of the graph of r: we have

$$T = \{(\xi(y), \xi(ry)) \ : \ y \in Y\} \subset X \times X. \tag{3.4}$$

In fact, the action of D_{2m} on Y packages the same data as the pair (X,T); the latter can be recovered by setting $X = Y/f$, defining $\xi : Y \to X$ to be the quotient map, and using equation (3.4) to define T.

Dihedral triples. Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a holomorphic 1–form and let $T \subset X \times X$ be a dihedral correspondence of type m. Then T induces an endomorphism of $\mathrm{Jac}(X)$ and hence a map

$$T^* : \Omega(X) \to \Omega(X).$$

The fact that T is symmetric implies that T^* is self–adjoint and that $[T] \in \mathrm{End}(\mathrm{Jac}(X))$ is invariant under the Rosati involution (cf. [GH, Ch. 2.5], [BL, §5]).

We say (X,ω,T) is a *dihedral triple* of type m if:

1. $T^*\omega = \tau_m\omega$, and
2. We have $Y/D_{2m} \cong \mathbb{P}^1$.

(Recall $\tau_m = \zeta_m + \zeta_m^{-1}$).

The Galois cover $\boldsymbol{Y/\mathbb{P}^1}$. For a more complete picture, it is useful to introduce the degree $2m$ rational map

$$\widetilde{\pi} : Y \to Y/D_{2m} \cong \mathbb{P}^1.$$

We have a unique rational map $\pi : X \to \mathbb{P}^1$, of degree m, making the diagram

$$\begin{tikzcd} D_{2m} \rightsquigarrow Y \arrow[dd, "\widetilde{\pi}"] \arrow[dr, "\xi"] & \\ & X = Y/f \arrow[dl, "\pi"] \\ \mathbb{P}^1 & \end{tikzcd} \tag{3.5}$$

commute. Letting $(b_1, \ldots, b_{p+1})$ denote the critical values of $\widetilde{\pi}$, the branched covering $Y/\mathbb{P}^1$ determines a surjective monodromy map

$$\rho : \pi_1(\mathbb{P}^1 - \{b_1, \ldots, b_{p+1}\}) \to D_{2m}, \tag{3.6}$$

well–defined up to conjugacy. Conversely, ρ determines the commutative diagram above.

The dihedral locus $\boldsymbol{\Omega D_{m,n,p}}$. We now turn to the definition of the locus of dihedral forms of type (m, n). In view of applications to general polygons, we will work in a slightly broader setting.

A *dihedral type* is a triple of integers (m, n, p) such that: $p \geq 2$ is even, $m \geq 3$, and $1 \leq n \leq p-1$. When m is odd we require $n = p/2$. Our primary interest is in the case $p = 6$.

Fix a dihedral type (m, n, p), and consider a commutative diagram of the form (3.5) determined by a representation ρ as in (3.6). We say $\pi : X \to \mathbb{P}^1$ is a *dihedral cover* of type (m, n) if the branch points $(b_1, \ldots, b_{p+1})$ can be ordered so that conjugacy class of the monodromy on a peripheral loop γ_i about b_i satisfies

$$[\rho(\gamma_i)] = \begin{cases} [f] & \text{for } i = 1, \ldots, n; \\ [rf] & \text{for } i = n+1, \ldots, p; \text{ and} \\ [r^n] & \text{for } i = p+1. \end{cases} \tag{3.7}$$

Note that the branch point b_{p+1} is distinguished by the fact that its monodromy is a rotation, rather than a reflection. We refer to the preimages X^* and Y^* of b_{p+1} on X and Y as the *special fibers* of π and $\widetilde{\pi}$ respectively.

We say a dihedral triple (X, ω, T) is a of type (m, n, p) if the covering $\pi : X \to \mathbb{P}^1$ has this type, *and* the divisor of ω satisfies

$$(\omega) = v \cdot X^*. \tag{3.8}$$

The space of all forms arising from dihedral triples of this type will be denoted by $\Omega D_{m,n,p}$.

We emphasize that equation (3.8) is a *geometric condition* on the rational map $\pi : X \to \mathbb{P}^1$. It is this condition that makes the study of the space of dihedral forms $\Omega D_{m,n}$ more subtle than the study of the space of cyclic forms ΩZ_a; the latter are described using only topological conditions on π. For example the dimension of $\Omega D_{m,n}$ depends in a subtle way on the value of n; see Theorem 4.1.

Alternative definition of $\boldsymbol{\Omega D_{m,n,p}}$. One can specify a dihedral form of type (m,n,p) without reference to T. It suffices to give an action of D_{2m} on Y, check that the induced map $\pi : X = Y/f \to \mathbb{P}^1$ is a dihedral cover of type (m,n,p), show $(\omega) = v \cdot X^*$, and verify that

$$(r + r^{-1})^*(\omega) = \tau_m \cdot \omega.$$

Then (X, ω, T) is a dihedral triple for T defined by (3.4).

The dihedral locus $\boldsymbol{\Omega D_{m,n}}$. As we will see below, dihedral forms of type (m,n,p) are related to polygons with N sides, where $p = 2N-2$. Since we are primarily interested in quadrilaterals, we adopt the shorthand

$$\Omega D_{m,n} = \Omega D_{m,n,6}.$$

Numerical invariants. It is useful to factor the map $\widetilde{\pi} : Y \to \mathbb{P}^1$ through the hyperelliptic curve $C = Y/\langle r \rangle$, which is a degree two cover of $\mathbb{P}^1$ branched over $(b_1, \ldots, b_p)$. One can easily show, using Riemann–Hurwitz, that the genera of these curves satisfy

$$\begin{aligned} 2g(Y) - 2 &= m(p-2) - 2\gcd(m,n), \\ 2g(X) - 2 &= m(p-2)/2 - \gcd(m,n) - n, \quad \text{and} \\ 2g(C) - 2 &= p - 4. \end{aligned} \tag{3.9}$$

Let X_i denote the fiber of X over b_i. Then we also have

$$|X_i| = \begin{cases} |f\backslash D_{2m}/f| = (m+2)/2 & \text{for } i = 1, \ldots, n, \text{ and} \\ |rf\backslash D_{2m}/f| = m/2 & \text{for } i = n+1, \ldots, p \end{cases} \tag{3.10}$$

when m is even; and $|X_i| = (m+1)/2$ for $i = 1, \ldots, p$ when m is odd. For the special fiber we have

$$|X^*| = \gcd(m,n)$$

in both cases. In particular, the dihedral forms lie in a single stratum: we have

$$\Omega D_{m,n,p} \subset \Omega\mathcal{M}_g(v^u), \tag{3.11}$$

where $g = g(X)$, $u = \gcd(m, n)$ and $v = (2g-2)/u$.

Cyclic forms and dihedral forms. There is a close relationship between dihedral forms and certain cyclic forms.

To describe this connection, let us say a cyclic type $a = (a_1, \ldots, a_N)$ and a dihedral type (m, n, p) *correspond* to one another if

$$\begin{aligned} 1 &= a_1 \leq \cdots \leq a_{N-1} \leq 2, \\ m &= (N-2)^{-1} \textstyle\sum_1^N a_i, \\ n &= a_1 + \cdots + a_{N-1}, \text{ and} \\ p &= 2N-2. \end{aligned} \tag{3.12}$$

Clearly a determines (m, n, p) and vice–versa. These conditions are consistent with equation (1.2); for examples of corresponding pairs with $N = 4$, see Table 1.

When a and (m, n, p) correspond as above, a dihedral form of type (m, n, p) can potentially degenerate to a cyclic form of type a, without changing the genus of X. We will see that such degenerations actually occur in Theorem 4.2.

Here is an explanation of the conditions above., For simplicity assume m is even. Let (X, ω, r) be a cyclic triple of type $(a_1, \ldots, a_N)$, with associated rational map $\pi : X \to \mathbb{P}^1 = X/r$ branched over $(b_1, \ldots, b_N)$. We wish to arrange that (1) the divisor of ω is supported on the special fiber X^* over b_N; and that (2) the map π can be deformed such that each branch point b_i, $i < N$, with cyclic monodromy, breaks up into a pair of points b_i', b_i'' with dihedral monodromy. The deformed map then has a total of $2N - 1 = p + 1$ branch points, as required for a dihedral map of type (m, n, p).

Condition (1) is equivalent to the condition that $a_i | m$ for all $i < N$, while condition (2) implies, by Riemann–Hurwitz, that $a_i = 1$ or 2. The case $a_i = 1$ arises when b_i splits into a pair of points with monodromy (f) and (rf), while $a_i = 2$ arises when the new points both have monodromy (f). Consequently the total number of points with monodromy (f) is given by $n = \sum_1^{N-1} a_i$. The monodromy around the final point b_N remains r^{a_N}, which lies in the conjugacy class $(r^n) \subset D_{2m}$ since $n + A_N = 0 \bmod m$. This explains the formula for n and the constraints on (a_i) in the system of equations (3.12), and the two different occurrence of n in equation (3.7) defining the monodromy of a dihedral cover.

Cohomology and dihedral forms. We now turn to the proof of Theorem 3.1.

Let (X, ω, T) be a dihedral triple of type (m, n, p). Thinking of T as a multivalued map, it is easy to see that $T(X^*) = X^*$. Thus T induces an automorphism of the relative cohomology group $H^1(X, X^*)$. On the level of forms, it is given by

$$T^*(\alpha) = \xi_*(r^* \xi^* \alpha). \tag{3.13}$$

Let

$$D(X, \omega, T) = \mathrm{Ker}(T^* - \tau_m) \subset H^1(X, X^*),$$

and let

$$\epsilon(m, n) = \begin{cases} 1 & \text{if } m \text{ divides } n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

We will prove Theorem 3.1 in the following more general form:

Theorem 3.3 *For any dihedral triple of type (m, n, p), we have*

$$\dim D(X, \omega, T) = p - 2.$$

The projection from $D(X, \omega, T)$ to $H^1(X)$ has fibers of dimension $2\epsilon(m, n)$.

Representations of D_{2m}. The proof rests on a description of the action of D_{2m} on $H^1(Y)$.

Let us first recall the characters of the irreducible representations of $\mathbb{Z}/m$ and D_{2m}. As in §2, the former will be denoted by χ_j, where $\chi_j(r) = \zeta_m^j$; and we say χ_j is primitive if $\gcd(j, m) = 1$.

Each 1–dimensional representation of $\mathbb{Z}/m$ induces a 2–dimensional representation of D_{2m} whose character $\widetilde{\chi}_j$ vanishes on reflections and satisfies $\widetilde{\chi}_j(r^i) = \zeta_m^{ij} + \zeta_m^{-ij}$ on rotations. This induced representation is irreducible unless $i = 0$ or $m/2$. We also have $\widetilde{\chi}_i = \widetilde{\chi}_{-i}$, so the number of 2–dimensional irreducible representations of D_{2m} is $\lfloor (m-1)/2 \rfloor$. In addition there are two 1–dimensional characters when m is odd, and four when m is even; they all satisfy $\chi(r) = \pm 1$.

A representation of D_{2m} is *primitive* if it is irreducible and it is induced from a primitive representation of $\mathbb{Z}/m$. The characters of these 2–dimensional representations are given by $\widetilde{\chi}_i$ with $\gcd(i, m) = 1$ and $m \geq 3$.

We are now in a position to prove:

Proposition 3.4 *Each primitive representation of D_{2m} occurs in $H^1(Y)$ with multiplicity $2g(C) - 2\epsilon(m, n)$.*

Proof. The argument is similar to proof of Theorem 2.5 on cyclic covers. For each $g \in D_{2m}$, let $\chi_Y(g) = \mathrm{Tr}(g|H^1(Y))$, and let $\mathrm{Fix}(g)$ denote the set of fixed–points of g acting on Y. By the Lefschetz trace formula, we have

$$\chi_Y(r^i) = 2 - |\,\mathrm{Fix}(r^i)|,$$

where we have adopted the convention that

$$|\,\mathrm{Fix}(r^0)| = \chi(Y) = 2|X^*| - 2mg(C).$$

For $i \neq 0$ in $\mathbb{Z}/m$, the fixed points of r^i (if any) lie on the special fiber Y^*, and hence

$$|\,\mathrm{Fix}(r^i)| = |Y^*| = 2|X^*|$$

when $i|\gcd(m,n) = |X^*|$; otherwise, $|\,\mathrm{Fix}(r^i)| = 0$.

Now consider the character $\widetilde{\chi}_j$ of a typical primitive representation of D_{2m}. By Frobenius reciprocity, its multiplicity in χ_Y is given by

$$\mu \;=\; \langle \chi_Y, \widetilde{\chi}_j \rangle_{D_{2m}} = \langle \chi_Y, \chi_j \rangle_{\mathbb{Z}/m} = \frac{1}{m} \sum_{i=0}^{m-1} (2 - |\,\mathrm{Fix}(r^i)|)\, \zeta_m^{ij}.$$

Since ζ_m^j is a primitive mth root of unity, we obtain

$$\mu = 2g(C) - \frac{1}{m} \sum_{i|\gcd(m,n)} 2|X^*|\zeta_m^{ij}.$$

The sum above is zero unless $m|n$, in which case it contributes -2 to the value of μ. In the latter case $\epsilon(m,n) = 1$, and the theorem follows. ■

Proposition 3.5 *Each primitive representation of D_{2m} occurs in $H^0(Y^*)$ with multiplicity $2\epsilon(m,n)$.*

Proof. Note that D_{2m} acts transitively on Y^*. If $|Y^*| = 2m$, then $H^0(Y^*)$ gives the regular representation of D_{2m}, and hence each primitive representation occurs with multiplicity equal to its dimension, which is $2 = 2\epsilon(m,n)$. Otherwise the action of D_{2m} on Y^* factors through a smaller dihedral group, so the primitive representations occur with multiplicity $\epsilon(m,n) = 0$. ■

Corollary 3.6 *Each primitive representation of D_{2m} occurs in $H^1(Y, Y^*)$ with multiplicity $2g(C)$.*

Proof. Apply the preceding two results to the exact sequence of $\mathbb{Z}[D_{2m}]$ modules

$$\mathbb{C} \to H^0(Y^*) \to H^1(Y, Y^*) \to H^1(Y) \to 0.$$

■

Eigenspaces and multiplicities. To complete the proof of Theorem 3.3, we note that $r + 1/r$ is in the center of $\mathbb{Z}[D_{2m}]$, and thus it acts by scalar multiplication on any irreducible $\mathbb{Z}[D_{2m}]$–module V. This scalar is given by ± 2 if $\dim(V) = 1$, and by

$$\lambda_j = \zeta_m^j + \zeta_m^{-j}$$

if $\dim(V) = 2$ and V has character $\widetilde{\chi}_j$. In the latter case the f–invariant subspace satisfies $\dim V^f = 1$. Consequently, if W is any finite–dimensional representation of D_{2m}, defined over $\mathbb{Q}$, and each primitive representation of D_{2m} occurs in W with multiplicity μ, then

$$\dim \operatorname{Ker}(r + r^{-1} - \tau_m)^f = \mu$$

as well.

Proof of Theorem 3.3. Applying the preceding reasoning with $W = H^1(Y, Y^*)$ and $W^f = H^1(X, X^*)$, we find that

$$\dim D(X, \omega, T) = \dim \operatorname{Ker}(r + 1/r - \tau_m)^f = 2g(C) = p - 2$$

by Proposition 3.6. Similarly, for $W = H^1(X)$ we find

$$\dim \operatorname{Ker}(r + 1/r - \tau_m)^f = 2g(C) - 2\epsilon(m, n)$$

by Proposition 3.4, and the theorem follows. ■

Proof of Theorem 3.1. Specialize the preceding result to the case $p = 6$, and observe that the condition $m \neq n$ implies $\epsilon(m, n) = 0$ and $Z(\omega) = X^*$, since $g(X) \geq 2$. ■

Notes. The endomorphism $T \in \operatorname{Jac}(X)$ determined by a dihedral triple (X, ω, T) is of Hecke type in the sense of [El] and [Wr1]. For a different perspective on dihedral covers of $\mathbb{P}^1$, see [CLP].

4 Equations for dihedral forms

In this section we return to our main topic, the case $p = 6$, and calculate the dimension of the variety

$$\Omega D_{m,n} = \Omega D_{m,n,6} \subset \Omega\mathcal{M}_g.$$

Here $2g - 2 = 2m - n - \gcd(m, n)$, and as in §3, we assume that $m \geq 3$, $0 < n < 6$, and $n = 3$ if m is odd.

The main result of this section is:

Theorem 4.1 *For $n \geq 3$, the space of dihedral forms $\Omega D_{m,n}$ is a nonempty, irreducible, unirational variety. For $m \leq 2g - 2$, its dimension is given by*

$$\dim(\Omega D_{m,n}) = 3 + \lfloor 2n/m \rfloor. \tag{4.1}$$

For $n < 3$, $\Omega D_{m,n}$ is empty.

Here $\lfloor q \rfloor$ is the largest integer $\leq q$.

Whenever (m, n) corresponds to a quadrilateral type $a = (a_1, a_2, a_3, a_4)$ via equation (1.2), we also show:

Theorem 4.2 *The closure of $\Omega D_{m,n}$ in $\Omega\mathcal{M}_g$ contains the corresponding variety of cyclic forms ΩZ_a.*

(This result holds for $(m, n) = (3, 3)$ and $(4, 4)$ as well, provided we take $a = (1, 1, 1)$ and $(1, 1, 2)$ respectively.)

Throughout this section, we use $\mathbb{C}(Z)$ to denote the field of rational functions on a compact Riemann surface Z.

Polynomials. To prove Theorem 4.1, we start by giving explicit algebraic equations for all dihedral triples of type (m, n).

Let $\mathbb{C}[x]_k$ denote the space of polynomials in x of degree k or less. We define, for each dihedral type (m, n), a subvariety

$$P_{m,n} \subset \mathbb{C}[x]_n \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor} \tag{4.2}$$

as follows: for m odd, we have $n = 3$, and we let

$$P_{m,n} = \left\{(p, q) : D = p^2 - 4q^m \text{ is separable of degree } 6\right\}; \tag{4.3}$$

while for m even, we have a factorization

$$D = p^2 - 4q^m = (p - 2q^{m/2})(p + 2q^{m/2}) = (d_1 s_1^2)(d_2 s_2^2) \tag{4.4}$$

where d_1 and d_2 are square free, and we let

$$P_{m,n} = \left\{ (p,q) : \begin{array}{c} d = d_1 d_2 \text{ is separable of degree } 6 \\ \text{and } \deg(d_1) = n \end{array} \right\}. \tag{4.5}$$

Note that equation (4.2) means that for all (p,q) in $P_{m,n}$, we have

$$\deg(p) \le n \quad \text{and} \quad \deg(q) \le \lfloor 2n/m \rfloor. \tag{4.6}$$

Consequently $\deg(D) \le 2n$ and $\deg(s_1) = 0$, i.e. s_1 is a nonzero constant.

Equations for $\boldsymbol{Y}$. Each pair of polynomials $(p,q) \in P_{m,n}$ determines a branched covering space $\widetilde{\pi} : Y \to \mathbb{P}^1$ with Galois group D_{2m}. The compact Riemann surface Y is defined by the polynomial equation

$$y^{2m} - p(x) y^m + q(x)^m = 0, \tag{4.7}$$

the branched covering is given by $\widetilde{\pi}(y,x) = x$, and the action of D_{2m} on Y is given by

$$r(x,y) = (x, \zeta_m y) \quad \text{and} \quad f(x,y) = (x, q/y).$$

This action is faithful, and the quotient Y/D_{2m} is the projective line $\mathbb{P}^1_x$ with coordinate x.

Equations for $\boldsymbol{X}$. As in §3 we then have a quotient map

$$\xi : Y \to X = Y/f,$$

and a degree m map $\pi : X \to \mathbb{P}^1$, such that $\widetilde{\pi} = \pi \circ \xi$.

Recall that the Chebyshev polynomial T_m of degree m is characterized by the equation

$$T_m((x + x^{-1})/2) = (x^m + x^{-m})/2.$$

(The map $T_m : \mathbb{P}^1 \to \mathbb{P}^1$ is itself a dihedral covering; see Appendix A). Rewriting (4.7) as

$$y^m + \frac{q^m}{y^m} = p,$$

and setting $u = (y + q/y)/2$, we find that the Riemann surface $X = Y/f$ is defined by the polynomial equation

$$2\, q^{m/2}\, T_m(q^{-1/2} u) \;=\; p. \tag{4.8}$$

The left hand side is a polynomial in q because $T_m(-x) = (-1)^m T(x)$. The dihedral map $\pi : X \to \mathbb{P}^1_x$ is given by $\pi(x,u) = x$.

The correspondence T. As remarked in §3, the D_{2m}-action on Y determines a dihedral correspondence

$$T \subset X \times X$$

by taking the image of the graph of r in $Y \times Y$.

Equations for C. Setting $z = y^m$ in (4.7), it is immediate that a defining equation for the genus two curve $C = Y/r$ is given by

$$z^2 - pz + q^m = 0, \tag{4.9}$$

a quadratic equation in z with discriminant $D = p^2 - 4q^m$. This explains the occurrence of D and its factorization in the definition of $P_{m,n}$. Indeed, the odd order zeros of D are the finite branch points $(b_1, \ldots, b_6)$ of the dihedral map $\widetilde{\pi} : X \to \mathbb{P}^1$, while the special fiber X^* lies over $b_7 = \infty$. When m is even, the discriminant factors as $D = (p - 2q^{m/2})(p + 2q^{m/2})$, and we will see (in the course of the proof of Proposition 4.11) that the zeros of the first factor with odd multiplicity gives branch points with monodromy $[f]$; thus we must have $n = \deg(d_1)$ to get a dihedral cover of type (m, n).

Equations for forms. Each $(p, q) \in P_{m,n}$ also determines a holomorphic 1–form $\nu \in \Omega(Y)$, defined by

$$\nu = y \cdot \frac{dx}{2y^m - p(x)}. \tag{4.10}$$

Let $\xi_* : \Omega(Y) \to \Omega(X)$ denote the pushforward map, and let

$$(X, \omega) = (Y, \nu)/f = (Y/f, \xi_*(\nu)) \in \Omega\mathcal{M}_g. \tag{4.11}$$

The form ω is given explicitly by

$$\omega = \frac{dx}{q^{(m-1)/2} U_{m-1}(q^{-1/2} u)} \tag{4.12}$$

where U_{m-1} is the Chebyshev polynomial of the second kind, of degree $m-1$; it satisfies

$$U_{m-1}\big((x + x^{-1})/2\big) = \frac{x^m - x^{-m}}{x - x^{-1}}.$$

Summing up, for each $(p, q) \in P_{m,n}$ we obtain a triple (X, ω, T) where $(X, \omega) \in \Omega\mathcal{M}_g$ and $T \subset X \times X$. The main step in the proof of Theorem 4.1 is to show:

Theorem 4.3 *For all $(p,q) \in P_{m,n}$, the associated triple (X,ω,T) is dihedral of type (m,n), and every dihedral triple of type (m,n) arises in this way.*

In particular, we have a surjective algebraic map

$$P_{m,n} \to \Omega D_{m,n}.$$

The unirationality of $\Omega D_{m,n}$ stated in Theorem 4.1 then follows from unirationality of $P_{m,n}$ (Proposition 4.15) when $n \geq 3$, and the fact that $P_{m,n} = \emptyset$ for $n < 3$ implies the same for the space of dihedral forms (Proposition 4.16).

Dimension. Now assume $n \geq 3$. To compute the dimension of $\Omega D_{m,n}$, we introduce the variety

$$V_{m,n} \subset \mathcal{M}_{0,7} \tag{4.13}$$

which records the positions of the critical values $(b_1,\ldots,b_7)$ of the dihedral maps $\pi : X \to \mathbb{P}^1$ associated to forms in $\Omega D_{m,n}$. (In the case $m = n$, $b_7 = \infty$ is actually not a critical value; it is defined as the image of X^*.) Using Theorem 4.3, we describe $V_{m,n}$ explicitly and prove:

Theorem 4.4 *For any dihedral type (m,n) with $n \geq 3$, the associated moduli space of critical values satisfies*

$$\dim(V_{m,n}) = 2 + \lfloor 2n/m \rfloor.$$

The proof of Theorem 4.1 is then completed by:

Theorem 4.5 *If, in addition, $m \leq 2g-2$, then the dimensions of $\Omega D_{m,n}$ and $V_{m,n}$ are related by*

$$\dim(\Omega D_{m,n}) = \dim(V_{m,n}) + 1.$$

The bound

$$m = \deg(\pi) \leq 2g - 2 = \deg(K_X)$$

is used to relate π to the canonical linear system on X.

Outline of this section. To establish the results above, we will proceed in three steps.

I. First, we will show that (a) the triple (X,ω,T) associated to $(p,q) \in P_{m,n}$ is dihedral of type (m,n), and (b) all dihedral triples arise in this way.

II. Then, we will check that $P_{m,n}$, and hence $\Omega D_{m,n}$, is unirational for $n \geq 3$ and empty for $n < 3$.

III. Finally, we will compute the dimension of $\Omega D_{m,n}$.

Step I (a): From polynomials to forms. Our first goal is to prove:

Proposition 4.6 *For every $(p, q) \in P_{m,n}$, the associated triple (X, ω, T) is dihedral of type (m, n).*

Fix $(p, q) \in P_{m,n}$. We begin the proof with following useful fact.

Proposition 4.7 *We have* $\gcd(p, q) = 1$ *in* $\mathbb{C}[x]$.

Proof. We just need to show p and q have no common zeros. For m odd, the definition of $P_{m,n}$ requires that $D = p^2 - 4q^m$ has only simple zeros, so we are done. Now suppose $m \geq 4$ is even, and $q(x_0) = p(x_0) = 0$. By the definition of $P_{m,n}$, the polynomial $d_1 s_1^2 = p - 2q^{m/2}$ has simple roots (i.e. s_1 is constant), so both p and d_1 have a simple roots at x_0. But then $d_2 s_2^2 = p + 2q^{m/2}$ also has a simple root at x_0, so d_2 also has a simple root there, contradicting our requirement $d_1 d_2$ is separable. ∎

Divisors on C. Let $C/\mathbb{P}^1_x$ be the curve defined by equation (4.9), namely $z^2 - p(x)z + q(x)^m = 0$. The rational function $x \in \mathbb{C}(C)$ presents C as a 2-fold covering branched over the zeros of $D = p^2 - 4q^m$ with odd multiplicity, i.e. the zeros of $d = d_1 d_2$. We have $\deg(d) = 6$ by the definition of $P_{m,n}$; thus C is irreducible, of genus 2, and its fiber C_∞ over $x = \infty$ consists of two distinct points.

Proposition 4.8 *The rational function $z \in \mathbb{C}(C)$ has polar divisor $(z)_\infty = nP + n'P'$, where $n' \leq n$ and $C_\infty = \{P, P'\}$.*

Proof. By the degree bounds (4.6), we have $\deg(p) \leq n$ and $\deg(q^m) \leq 2n$; thus $\deg(D) \leq 2n$, where $D = p^2 - 4q^m$ is the discriminant of (4.9). By the quadratic formula, $z = (p \pm \sqrt{D})/2$; since p, q and D are polynomials in x, we must have $(z)_\infty = aP + bP'$ for some a, b with $n \geq a \geq b$. Now observe that $a = n$ whenever $\deg(p) = n$ or $\deg(D) = 2n$. This is automatic when m is odd, because $\deg(D) = 2n = 6$ by the definition of $P_{m,n}$. In the even case, use the fact that $\deg(d_1) = \deg(p - 2q^{m/2}) = n$. ∎

Proposition 4.9 *Every zero of z outside of C_∞ has order divisible by m.*

Proof. Let $Q \in C - C_\infty$ be a zero of z projecting to $R \in \mathbb{P}^1_x$, and let Q' be its image under the hyperelliptic involution. Since $\gcd(p,q) = 1$, and $q(R) = 0$, we have $p(R) \neq 0$ and hence $D(R) \neq 0$. Thus $Q \neq Q'$. We also have

$$\operatorname{ord}(z,Q) + \operatorname{ord}(z,Q') = \operatorname{ord}(q^m, R).$$

Since $\gcd(p,q) = 1$ by Proposition 4.7, $p(R) = z(Q) + z(Q') = z(Q') \neq 0$, and hence $\operatorname{ord}(z,Q) = m \operatorname{ord}(q,R)$. ∎

Proposition 4.10 *The curves Y and T are irreducible.*

Proof. Let us first recall the following fact about curves of genus two: if $w \in \mathbb{C}(C)$ and $(w) + 2C_\infty \geq 0$, then $w \in \mathbb{C}[x]$. (For the proof, note that every quadratic polynomial $w = w(x)$ satisfies $(w) + 2C_\infty \geq 0$, the space of quadratic polynomials is 3–dimensional, and $h^0(2C_\infty) = 3$ by Riemann–Roch.)

Now let us turn to irreducibility of Y. It suffices to show there is no $w \in \mathbb{C}(C)$ such that $w^k = z$ and $k > 1$, for then the polynomial $y^m = z$ defining Y is irreducible over $\mathbb{C}(C)$.

Suppose to the contrary we have $w^k = z$. Then k divides n, since $\operatorname{ord}(z,P) = -n$ by Proposition 4.8. The same Proposition shows that $(z) + nC_\infty \geq 0$, and hence

$$(w) + (n/k)C_\infty \geq 0.$$

Since $n \leq 5$ and $k \geq 2$, we have $n/k \leq 2$, and hence $w \in \mathbb{C}[x]$ by our initial remarks, contradicting the fact that $z = w^k$ generates $\mathbb{C}(C)$ over $\mathbb{C}(x)$.

The curve T is also irreducible since it is the image of Y under a morphism to $X \times X$. ∎

Monodromy of $Y/\mathbb{P}^1_x$. Next we analyze the monodromy of the map $\widetilde{\pi} : Y \to \mathbb{P}^1_x$; equivalently, we describe the set of fixed points $\operatorname{Fix}(g) \subset Y$ for various $g \in D_{2m}$.

Let $(b_1, \ldots, b_6)$ denote the zeros of the separable polynomial $d(x)$ vanishing at the odd order zeros of $D = p^2 - 4q^m$; and let $b_7 = \infty$. As we have seen:

The map $C \to \mathbb{P}^1_x$ is a degree two covering, branched over $(b_1, \ldots, b_6)$.

Proposition 4.9 implies:

The map $Y \to C$ is unramified outside $C_\infty = \{P, P'\}$.

Since $C = Y/r$, we find:

The set $\mathrm{Fix}(r^k) \subset Y$ *lies over* b_7 *for* $0 < k < m$.

Note that the reflections in D_{2m} acting on Y cover the hyperelliptic involution on C. Consequently, their fixed points map to Weierstrass points of C. This shows:

The set $\mathrm{Fix}(f) \cup \mathrm{Fix}(rf) \subset Y$ *lies over* $(b_1, \ldots, b_6)$ *on* $\mathbb{P}^1_x$.

Now suppose m is even, so we have the factorization

$$D = p^2 - 4q^m = (p - 2q^{m/2})(p + 2q^{m/2}) = (d_1 s_1^2)(d_2 s_2^2)$$

(equation (4.4)). Recall that $d = d_1 d_2$ is separable and $\deg(d_1) = n$ by the definition of $P_{m,n}$. Thus we can order the zeros of $d(x)$ so that $(b_1, \ldots, b_n)$ are zeros of d_1. We claim:

When m is even, $\mathrm{Fix}(f)$ *lies over* $(b_1, \ldots, b_n)$ *and* $\mathrm{Fix}(rf)$ *lies over* $(b_{n+1}, \ldots, b_6)$.

Indeed, at a fixed point of f we have $y^2 = q$, which together with the defining equation (4.7) for Y gives

$$0 = q^m - pq^{m/2} + q^m = q^{m/2}(2q^{m/2} - p) = -q^{m/2}(d_1 s_1^2). \tag{4.14}$$

We also have $D = d = 0$. Since $\gcd(p, q) = 1$, this implies that $q \neq 0$ and hence $d_1 = 0$. Similar reasoning applies to $\mathrm{Fix}(rf)$.

We can now show:

Proposition 4.11 *The map $\pi : X \to \mathbb{P}^1_x$ is dihedral of type (m, n) with special fiber $X^* = X_\infty$.*

Proof. It suffices to show that Y itself has the required monodromy over $(b_1, \ldots, b_7)$. Let $g_i = \rho(\gamma_i) \in D_{2m}$ denote monodromy of Y for a peripheral loop around b_i, as in §3. For $i \neq 7$, the map $Y \to \mathbb{P}^1_x$ is ramified over b_i, since C is; thus $[g_i] \neq \mathrm{id}$. Since $Y \to C = Y/r$ is ramified only over b_7, $[g_i]$ is a reflection. Our observations above then show:

$$[g_i] = [f] \text{ for } 1 \leq i \leq n, \text{ and } [g_i] = [rf] \text{ for } n + 1 \leq i \leq 6.$$

It remains only to show that $[g_7] = [r^n]$. This follows readily from the fact that $y^m = z$ and z has a pole of order n at $P \in C_\infty$ (Proposition 4.8). ∎

The dihedral form. We can now show the form (X, ω) determined by $(p, q) \in \mathbb{P}_{m,n}$ via equation (4.11) is dihedral of type (m, n).

Let

$$\eta = \frac{dx}{2z - p(x)} \ \in \ \Omega(C)$$

be a 1–form with $(\eta) = P + P' = C^* = C_\infty$, and let $\widetilde{\eta}$ denote its pullback to Y. Consider the meromorphic forms ν and ω on Y and X defined by $\nu = y \cdot \widetilde{\eta}$ as in equation (4.10), and $\omega = \xi_*(\nu)$ as in equation (4.11).

Proposition 4.12 *The divisor of ω is a non-negative multiple of X^*, and $T^*(\omega) = \tau_m \omega$.*

Proof. We first check that $\omega \neq 0$. To see this, note that ν and $f^*(\nu)$ are r–eigenforms on Y with eigenvalues ζ_m and ζ_m^{-1} respectively. Since ζ_m is a primitive mth root of unity, with $m \geq 3$, we have $\zeta_m \neq \zeta_m^{-1}$ and hence these eigenforms are linearly independent. This shows that

$$\xi^*(\omega) = \nu + f^*(\nu) \neq 0,$$

and hence $\omega \neq 0$.

Now it is routine to check that

$$(\widetilde{\eta}) = \frac{2m - \gcd(m,n)}{\gcd(m,n)} \cdot Y^*.$$

By Proposition 4.8 the divisor $(z) + nC^*$ is effective, so the same is true of the divisor $(y) + n/\gcd(m,n) \cdot Y^*$. It follows that

$$(\nu) = (y) + (\widetilde{\eta}) \geq v \cdot Y^*,$$

where

$$v = \frac{2m - \gcd(m,n) - n}{\gcd(m,n)} = \frac{2g(X) - 2}{|X^*|} \geq 0.$$

In particular, $\nu \in \Omega(Y)$ and hence $\omega \in \Omega(X)$. Since the degree two map $\xi : Y \to X$ is unramified over X^*, we have $(\omega) \geq v \cdot X^*$. But $v \cdot X^*$ has the degree of a canonical divisor on X, so equality must hold.

Finally, we compute $T^*(\omega)$ using equation (3.13): we have

$$T^*(\omega) = \xi_* r^* \xi^*(\omega) = \xi_* r^*(\nu + f^*(\nu)) = \xi_*(\zeta_m \nu + \zeta_m^{-1} f^*(\nu)) = \tau_m \omega, \quad (4.15)$$

as required for a dihedral form. ∎

Proof of Proposition 4.6. Combine Propositions 4.11 and 4.12. ∎

Remark: Hyperelliptic quotients. When m is even, D_{2m} contains three subgroups of index two, namely $H_0 = \langle r\rangle$, $H_1 = \langle r^2, rf\rangle$ and $H_2 = \langle r^2, f\rangle$. Thus the map $Y \to \mathbb{P}^1_x$ can be factored through three different hyperelliptic curves, namely $C_i = Y/H_i$ for $i = 0, 1, 2$. We have $C_0 = C$, while C_i is defined by the equation $z_i^2 = d_i(x)$ for $i = 1, 2$. The analysis of the fixed points of f and rf acting on Y can be reduced to a study of the Weierstrass points of C_1 and C_2.

Step I (b): From forms to polynomials. Our next task is to show:

Proposition 4.13 *Every dihedral triple* (X, ω, T) *of type* (m, n) *arises from a pair of polynomials* $(p, q) \in P_{m,n}$.

The key point in the argument is to use the existence of the 1-form ω, vanishing only on X^*, to produce a *polynomial* equation of the form (4.7) for Y, with $\deg(p)$ and $\deg(q)$ *relatively small.*

Notation and normalizations. To set the stage for the proof, we begin by elaborating several properties of dihedral forms.

Let (X, ω, T) be an arbitrary dihedral triple of type (m, n), let Y be the normalization of T, let $\pi : X \to \mathbb{P}^1_x$ be the associated rational map with dihedral monodromy, and set $C = Y/r$. Let $\mathbb{C}(x) = \mathbb{C}(\mathbb{P}^1_x)$ denote the function field of the target of π.

After changing coordinates on $\mathbb{P}^1_x$, we can assume that the special fibers all lie over the point $x = \infty$. We will consider the special fibers X^*, Y^* and C^*, as well as the fixed–point set $\mathrm{Fix}(f) \subset Y$, as divisors with multiplicity one at each point.

Forms on Y. Let

$$\widetilde{\omega} = \xi^*(\omega) \in \Omega(Y).$$

For the proof of Proposition 4.13, we will begin by constructing an r–invariant form $\widetilde{\eta} \in \Omega(Y)$ whose divisor is supported on Y^*. Then, we will write

$$\widetilde{\omega} = \nu + f^*(\nu),$$

for a suitable r–eigenform $\nu \in \Omega(Y)$; and finally, we will show that

$$y = \nu/\widetilde{\eta} \in \mathbb{C}(Y)$$

satisfies an equation of the form (4.7).

Recall from §3 that $(\omega) = v \cdot X^*$, where

$$v = \frac{2m - n - \gcd(m,n)}{\gcd(m,n)} = \frac{|\chi(X)|}{|X^*|}.$$

Since the critical points of $\xi : Y \to X$ coincide with $\mathrm{Fix}(f)$, we have

$$(\widetilde{\omega}) = \mathrm{Fix}(f) + v \cdot Y^*. \tag{4.16}$$

Let $\eta \in \Omega(C)$ be a form with $(\eta) = C^*$, and let $\widetilde{\eta}$ be its pullback to Y. Since $Y \to C$ is branched to order $m/\gcd(m,n)$ at each point in Y^*, and unbranched otherwise, we have

$$(\widetilde{\eta}) = \frac{2m - \gcd(m,n)}{\gcd(m,n)} \cdot Y^*. \tag{4.17}$$

Next we construct an r-eigenform ν associated to $\widetilde{\omega}$. From the definition of a dihedral triple, we have

$$T^* \omega = \tau_m \omega$$

Similarly, we have $(r + r^{-1})^* \widetilde{\omega} = \tau_m \widetilde{\omega}$.

Proposition 4.14 *The 1-form*

$$\nu = \frac{r^* \widetilde{\omega} - \zeta_m^{-1} \widetilde{\omega}}{\zeta_m - \zeta_m^{-1}} \in \Omega(Y) \tag{4.18}$$

is a ζ_m-eigenform for r with $\xi_*(\nu) = \omega$ and $(\nu) \geq v \cdot Y^*$.

Proof. Since $(r + r^{-1})^* \widetilde{\omega} = (\zeta_m + \zeta_m^{-1}) \widetilde{\omega}$, we have

$$\nu = \frac{\zeta_m \widetilde{\omega} - (r^{-1})^* \widetilde{\omega}}{\zeta_m - \zeta_m^{-1}},$$

implying that $r^*(\nu) = \zeta_m \nu$. From $fr = r^{-1} f$, we compute that

$$f^*(\nu) + \nu = \widetilde{\omega},$$

hence $\xi_*(\nu) = \omega$. The claim about the divisor of ν follows from the fact that ν is a linear combination of $\widetilde{\omega}$ and $r^*(\widetilde{\omega})$, both of which vanish to order v on $Y^* = r(Y^*)$. ∎

Eigenfunctions for r. Now form the rational function

$$y = \nu/\widetilde{\eta} \in \mathbb{C}(Y). \tag{4.19}$$

Combining $(\nu) \geq v \cdot Y^*$ (Proposition 4.14) with (4.17), we have that

$$(y) + \frac{n}{\gcd(m,n)} \cdot Y^* \geq 0 \quad \text{and} \quad \deg(y)_\infty \leq 2n. \tag{4.20}$$

Since r fixes $\widetilde{\eta}$, and ν is a primitive r-eigenform, y is a primitive r-eigenfunction. In particular, y generates $\mathbb{C}(Y)/\mathbb{C}(x)$, i.e.

$$\mathbb{C}(x,y) = \mathbb{C}(Y). \tag{4.21}$$

To see this, observe that by Galois theory we can write $\mathbb{C}(y,x) = \mathbb{C}(Y)^H$ for some subgroup $H \subset D_{2m}$. Since y is a primitive r–eigenfunction, we have $H \cap \langle r \rangle = \{\mathrm{id}\}$, and thus either H is trivial or $H = \langle r^i f \rangle$ for some i. But $\mathbb{C}(y,x)$ is r–invariant, so $rHr^{-1} = H$; hence H is trivial and equation (4.21) follows.

The defining equation for Y. Let

$$p = y^m + f^*(y^m) \quad \text{and} \quad q = yf^*(y) \tag{4.22}$$

in $\mathbb{C}(Y)$. The rational functions p, q are D_{2m}-invariant; hence they lie in $\mathbb{C}(x)$. In fact, from (4.20), we see that p, q are polynomials and

$$p, q \in \mathbb{C}[x], \quad \text{and} \quad \deg(p) \leq n, \deg(q) \leq 2n/m. \tag{4.23}$$

We claim the defining equation for Y is given by

$$y^{2m} - p(x)y^m + q(x)^m = 0,$$

in agreement with (4.7). To see this, note that y has degree $2m = [\mathbb{C}(Y) : \mathbb{C}(x)]$ over $\mathbb{C}(x)$ by (4.21), and y satisfies the claimed equation by (4.22). In view of equation (4.22), the action of D_{2m} on Y is given by $f(x,y) = (x, q/y)$ and $r(x,y) = (x, \zeta_m y)$.

Equations for ν and ω. Similarly, the form ν in $\Omega(Y)$ is given by

$$y \cdot \frac{dx}{2y^m - p(x)}$$

up to scale, in agreement with (4.10). To see this, just observe that $(dx/(2z - p(x)) = C^*$, and hence the expression above is proportional to $\nu = y\widetilde{\eta}$ (see equation (4.19)).

By the same token, the form (X, ω) agrees with the form defined by equation (4.11) up to scale, since $\omega = \xi_*(\nu)$ by Proposition 4.14.

Proof of Proposition 4.13. The above discussion shows that from an arbitrary dihedral triple (X, ω, T) of type (m, n), we can construct polynomials (p, q) as above so that (Y, ν) and (X, ω) are defined by the desired equations (4.7), (4.10) and (4.11) up to scale. But it is straightforward to check the set of forms in $\Omega\mathcal{M}_g$ coming from $P_{m,n}$ is closed under scalar multiplication. Thus it remains only to show that $(p, q) \in P_{m,n}$. Note that the degrees of p and q are correct by (4.23).

The argument is very similar to the discussion of the monodromy of $Y/\mathbb{P}^1_x$ in Step I(a). Let $D = p^2 - 4q^m$ be the discriminant of the definition equation $z^2 - pz + q^m = 0$ for C. Write $D = s^2 d$ where d has simple zeros. Since C has genus two (by equation (3.9)), we have $\deg(d) = 6$. When m is odd, this shows $(p, q) \in P_{m,n}$ as desired (see equation (4.3)).

In the even case, simply write $D = (d_1 s_1^2)(d_2 s_2^2)$ as in (4.4), and observe that $\deg(d_1) = n$ because the covering $\pi : X \to \mathbb{P}^1_x$ has type n fibers with monodromy $[f]$ (compare equation (4.14)). ■

Proof of Theorem 4.3. Combine Propositions 4.6 and 4.13. ■

Step II: Unirationality. Now that we have a surjective map from $P_{m,n}$ with $\Omega D_{m,n}$, unirationality (and irreducibility) of the latter variety follows from:

Proposition 4.15 *For $n \geq 3$, the space $P_{m,n}$ is a nonempty, unirational variety.*

Proof. For m odd, $P_{m,n}$ is nonempty because $(x^3, 1) \in P_{m,n}$, and is rational because it is defined by open conditions on the vector space $\mathbb{C}[x]_n \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor}$.

For m even, consider the variety

$$P'_{m,n} \subset \mathbb{C}[x]_{6-n} \times \mathbb{C}[x]_{n-3} \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor}$$

defined by

$$P'_{m,n} = \left\{ (d_2, s_2, q) : d_2(d_2 s_2^2 - 4q^{m/2}) \text{ is separable of degree } 6 \right\}. \quad (4.24)$$

Note that for the product on the left to have degree 6, we must have

$$\deg(d_2) = 6 - n \quad \text{and} \quad \deg(d_2 s_2^2 - 4q^{m/2}) = n; \quad (4.25)$$

in fact $d_1 = d_2 s_2^2 - 4q^{m/2}$. The space $P'_{m,n}$ is nonempty, since it contains (d_2, x^{n-3}, c) whenever d_2 is separable of degree $6-n$ and c is a generic constant. It is also a rational variety, since it is defined by a single open condition on $\mathbb{C}[x]_{6-n} \times \mathbb{C}[x]_{n-3} \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor}$.

To show $P_{m,n}$ is unirational, it suffices to show that the map $P'_{m,n} \to P_{m,n}$ defined by $(d_2, s_2, q) \mapsto (p, q)$, with $p = d_2 s_2^2 - 2q^{m/2}$, is surjective. But this is clear: for every $(p,q) \in P_{m,n}$, we can compute an appropriate (d_2, s_2) from the unique factorization of $p + 2q^{m/2}$ in $\mathbb{C}[x]$. ∎

Proposition 4.16 *For $n \leq 2$, the space $\Omega D_{m,n}$ is empty.*

Proof. If $n \leq 2$ then m is even, and the space $P'_{m,n}$ defined above is empty because there are no polynomials with $\deg(s_2) = n-3 < 0$. Hence $P_{m,n} = \emptyset$, and therefore $\Omega D_{m,n} = \emptyset$ by Proposition 4.13. ∎

We are now ready to prove that the closure of $\Omega D_{m,n}$ in $\Omega\mathcal{M}_g$ contains an associated variety of cyclic forms.

Proof of Theorem 4.2. For $(p,q) \in P_{m,n}$, the associated Riemann surface $X = Y/f$ is defined by equation (4.8) which is of the form

$$(2u)^m + O(q) = p.$$

By equation (4.12), the 1-form ω is given by

$$\omega = \frac{dx}{(2u)^{m-1} + O(q)}.$$

First suppose m is odd. In this case, $a = (1,1,1,2m-3)$. We take the limit as $q \to 0$ in $P_{m,n}$, which we can do since $P_{m,n}$ is open and dense in $\mathbb{C}[x]_n \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor}$. In this limit, we obtain the defining equations for a generic form in ΩZ_a.

Now suppose m is even. Then the quadrilateral type associated to (m,n) is $a = (1^{6-n}, 2^{n-3}, 2m-n)$. Taking the limit $q \to 0$ in $P'_{m,n}$, we obtain the equation for X

$$(2u)^m = p = d_2 s_2^2$$

where $\deg(d_2) \leq 6-n$ and $\deg(s_2) \leq n-3$. The form $\omega = dx/(2u)^{m-1}$ is generic form in ΩZ_a. ∎

Step III: Dimension of the variety of dihedral forms. Throughout this step we assume $n \geq 3$. Recall that $V_{m,n} \subset \mathcal{M}_{0,7}$ is the moduli space of branch point configurations $(b_1, \ldots, b_7)$ that arise from the dihedral maps attached to forms in $\Omega D_{m,n}$ (see equation (4.13)). We will show that

$$\dim(V_{m,n}) = 2 + \lfloor 2n/m \rfloor$$

and, provided $m \leq 2g-2$, we have

$$\dim(\Omega D_{m,n}) = \dim(V_{m,n}) + 1 = 3 + \lfloor 2n/m \rfloor.$$

These statements are Theorems 4.4 and 4.5 respectively.

Sums of polynomial powers. We will use the following lemma to determine the dimension of $V_{m,n}$.

Lemma 4.17 *Fix a nonzero polynomial $r \in \mathbb{C}[x]$ and integers $a, b \geq 2$. Then the map $\psi\colon \mathbb{C}[x]_d \times \mathbb{C}[x]_e \to \mathbb{C}[x]$ given by*

$$\psi(p,q) = rp^a + q^b$$

is generically an immersion, provided $\deg(r) > 0$ *when $a = b = 2$ or $d = 0$.*

Proof. The derivative of ψ is given by

$$\dot{\psi} = arp^{a-1}\dot{p} + bq^{b-1}\dot{q}.$$

To show ψ is an immersion at (p,q), it suffices to show that when $\dot{\psi} = 0$, we also have $\dot{p} = \dot{q} = 0$. We may assume that p and q have degrees d and e respectively, and $\gcd(rp, q) = 1$, since these conditions are generic.

Suppose, to the contrary, that $\dot{\psi} = 0$ but at least one of $\dot{p}$ and $\dot{q}$ is nonzero. Then the fact that $\gcd(pr, q) = 1$ implies $\dot{p}$ is a multiple of q^{b-1} and $\dot{q}$ is a multiple of rp^{a-1}. From this we find

$$\begin{aligned} d &\geq \deg(\dot{p}) = (b-1)\deg(q) = (b-1)e \geq (b-1)\deg(\dot{q}) \\ &= (b-1)((a-1)\deg(p) + \deg(r)) \geq (b-1)(a-1)d + \deg(r) \geq d, \end{aligned}$$

where we have used the fact that $a, b \geq 2$. Thus the last inequality is an equality. For equality to hold, we must have $a = b = 2$ or $d = 0$; and moreover, we must have $\deg(r) = 0$. But these cases are excluded by hypothesis. ∎

Proof of Theorem 4.4. We wish to show $\dim(V_{m,n}) = 2 + \lfloor 2n/m \rfloor$.

First suppose that m is odd. Then $n = 3$, and there is a rational map $\mathbb{C}[x]_6 \to \mathcal{M}_{0,7}$ sending a polynomial $d(x)$ to the configuration consisting of its zeros along with ∞. The fibers of this map are orbits of $\mathbb{C}^* \times \mathrm{Aut}(\mathbb{C})$ on $\mathbb{C}[x]_6$, where $\mathrm{Aut}(\mathbb{C})$ acts by precomposition and $\mathbb{C}^*$ acts by postcomposition by scaling.

The variety $V_{m,n}$ corresponds to the image of

$$\begin{gathered} \psi\colon \mathbb{C}[x]_n \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor} \to \mathbb{C}[x]_6 \\ \psi(p,q) = p^2 + q^m. \end{gathered} \tag{4.26}$$

To see this, we use Theorem 4.3 together with the fact that $P_{m,n}$ is open and dense in $\mathbb{C}[x]_n \times \mathbb{C}[x]_{\lfloor 2n/m \rfloor}$ when m is odd. By Lemma 4.17, ψ is generically an immersion, so its image has dimension $(n+1)+(\lfloor 2n/m \rfloor+1) = 5 + \lfloor 2n/m \rfloor$ and codimension $2 - \lfloor 2n/m \rfloor$. Since the image of ψ is invariant under $\mathbb{C}^* \times \mathrm{Aut}(\mathbb{C})$, $V_{m,n}$ also has codimension $2 - \lfloor 2n/m \rfloor$ in $\mathcal{M}_{0,7}$, hence

$$\dim(V_{m,n}) = \dim(\mathcal{M}_{0,7}) - 2 + \lfloor 2n/m \rfloor = 2 + \lfloor 2n/m \rfloor$$

as desired.

Now suppose that m is even. For this case we will use the variety $P'_{m,n}$ defined in equation (4.24). There is a rational map $\mathbb{C}[x]_n \times \mathbb{C}[x]_{6-n} \to \mathcal{M}_{0,7}$ sending a pair (d_1, d_2) to the configuration of zeros for $d = d_1 d_2$ along with ∞. The fibers of this map are $\mathrm{Aut}(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$-orbits in $\mathbb{C}[x]_n \times \mathbb{C}[x]_{6-n}$ ($\mathrm{Aut}(\mathbb{C})$ acts diagonally, and $\mathbb{C}^* \times \mathbb{C}^*$ acts by scaling each component).

By Theorem 4.3 together with the parameterization of $P'_{m,n} \to P_{m,n}$ in the proof of Proposition 4.15, the variety $V_{m,n}$ corresponds to the image of

$$\begin{gathered} \phi\colon P'_{m,n} \to \mathbb{C}[x]_n \times \mathbb{C}[x]_{6-n} \\ \phi(d_2, s_2, q) = (d_2 s_2^2 - 4q^{m/2}, d_2). \end{gathered} \tag{4.27}$$

Fix a separable $d_2 \in \mathbb{C}[x]_{6-n}$ with $\deg(d_2) = 6 - n$. (Equations (4.24) and (4.25) imply that any d_2 appearing as the first coordinate of a point in $P'_{m,n}$ has this type.) Applying Lemma 4.17 to the map $(s_2, q) \mapsto d_2 s_2^2 - 4q^{m/2}$, we conclude that the fiber of $P'_{m,n}$ over $d_2 \in \mathbb{C}[x]_{6-n}$ has dimension $n - 1 + \lfloor 2n/m \rfloor$, and codimension $2 - \lfloor 2n/m \rfloor$ in $\mathbb{C}[x]_n$. Hence $\psi(P'_{m,n})$ has codimension $2 - \lfloor 2n/m \rfloor$ in $\mathbb{C}[x]_n \times \mathbb{C}[x]_{6-n}$. Since $\psi(P'_{m,n})$ is invariant under $\mathrm{Aut}(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$, the variety $V_{m,n}$ also has codimension $2 - \lfloor 2n/m \rfloor$ in $\mathcal{M}_{0,7}$. Hence $V_{m,n}$ itself has dimension $2 + \lfloor 2n/m \rfloor$. ■

An incidence correspondence. To relate the dimensions of $\Omega D_{m,n}$ and $V_{m,n}$, we form the incidence correspondence

$$\Omega D'_{m,n} \subset \Omega D_{m,n} \times V_{m,n}$$

consisting of triples (X, ω, B) where $(X, \omega) \in \Omega D_{m,n}$ and an associated dihedral map $\pi\colon X \to \mathbb{P}^1$ has critical configuration $B \in V_{m,n}$.

Proposition 4.18 *Each fiber of $\Omega D'_{m,n} \to V_{m,n}$ is a finite union of lines.*

Proof. Each critical configuration $B \in \mathcal{M}_{0,7}$ determines $\pi\colon X \to \mathbb{P}^1$ and X^* up to finitely many choices, and (X, X^*) determines ω up to scale. Hence each fiber of $\Omega D'_{m,n} \to V_{m,n}$ is a finite union of lines. ∎

Proposition 4.19 *If $m \leq 2g-2$, the fibers of $\Omega D'_{m,n} \to \Omega D_{m,n}$ are finite.*

Proof. Let (X, ω, T) be a dihedral triple of type (m, n), with associated dihedral map $\pi\colon X \to \mathbb{P}^1_x$. Let B denote the configuration of critical values of π, so that $(X, \omega, B) \in \Omega D'_{m,n}$. The condition $m \leq 2g-2$ ensures that π comes from a linear subsystem of the canonical system on X. We claim that, under this hypothesis, the map π is determined by the Jacobian endomorphism induced by T. Fixing $(X, \omega) \in \Omega D_{m,n}$, there are countably many endomorphisms of $\mathrm{Jac}(X)$, hence countably many choices for π and B. Since $\Omega D'_{m,n} \to \Omega D_{m,n}$ is algebraic with countable fibers, its fibers are in fact finite.

We now prove our claim that T^* determines B. Let

$$W = \ker(T^* - \tau_m)|_{\Omega(X)}.$$

Since T is holomorphic, the eigenspaces for T^* are compatible with the Hodge structure. More precisely, the projection of $D(X, \omega, T)$ to $H^1(X)$ is the direct sum of W and its complex conjugate. Using Theorem 3.1, we have that $\dim(W) = \dim(D(X, \omega, T))/2 = 2$ (note that $(m, n) \neq (3, 3)$ since $m \leq 2g-2$).

Clearly, we have that $\omega \in W$. The form $x\omega$ has divisor

$$(x\omega) = (x) + (\omega) \geq (v - m/\gcd(m, n)) \cdot X^*$$

where, as usual, $v = (2m - 2g + 2)/\gcd(m, n)$ is the order of the zeros of ω. The bound $m \leq 2g-2$ ensures that $v - m/\gcd(m, n) \geq 0$, and $x\omega$

holomorphic. We also have that $x\omega \in W$. To see this, note that $T^*(x) = \xi_* r^* \xi^*(x) = x$ since the coordinate x on Y is D_{2m}-invariant. Hence

$$T^*(x\omega) = T^*(x)T^*(\omega) = \tau_m x\omega.$$

The space W is spanned by ω and $x\omega$, and the map to $\mathbb{P}^1$ determined by this linear system is equal to $\pi\colon X \to \mathbb{P}^1_x$.

Thus, the space W, which is determined by T^*, determines the map π, and the critical configuration B. ∎

Proof of Theorem 4.5. By Proposition 4.19, we have

$$\dim(\Omega D'_{m,n}) = \dim(V_{m,n}) + 1$$

for all (m,n), and $\dim(\Omega D_{m,n}) = \dim(\Omega D'_{m,n})$ provided $m \leq 2g-2$. Thus $\dim(\Omega D_{m,n}) = \dim(V_{m,n}) + 1$ under the same condition on m. ∎

Proof of Theorem 4.1. With steps I, II and III in place, we can now complete the proof of our main result. Indeed, for $n \geq 3$, the variety $\Omega D_{m,n}$ is nonempty, irreducible and unirational by Theorem 4.3 and Proposition 4.15, and $\dim(\Omega D_{m,n}) = 3 + \lfloor 2n/3 \rfloor$ provided $m \leq 2g-2$, by Theorems 4.4 and 4.5; while $\Omega D_{m,n} = \emptyset$ for $n < 3$, by Proposition 4.16. ∎

Remarks. We conclude with some comments on general dihedral covers, degrees, divisors and other topics around the discussion above.

1. *Relation to the canonical map.* We emphasize that when $m \leq 2g(X)-2$ (a bound that holds in most cases of interest, including the six examples given in Table 1), the dihedral map $\pi : X \to \mathbb{P}^1_x$ comes from the canonical linear system $|K_X|$. That is, we can factor this map as a composition
$$X \to \mathbb{P}^{g-1} \to \mathbb{P}^1_x$$
of the canonical embedding and a linear projection. (For a more complete discussion of the interplay between the canonical embedding and the dihedral map in the case $(m,n) = (6,3)$, see [MMW, §4].)

2. *Superabundance.* On the other hand, when $m > 2g-2$, the formula $\dim(\Omega D_{m,n}) = 3 + \lfloor 2n/m \rfloor$ can fail. For instance, when $(m,n) = (4,5)$, we have
$$3 + \lfloor 2n/m \rfloor = 5 > 4 \geq \dim \Omega D_{m,n}$$
by Theorem 3.1 (and in fact $\dim \Omega D_{m,n} = 4$). The preceding arguments do not apply because we need π to come from a subcanonical linear series in the proof of Proposition 4.19.

3. *General dihedral extensions.* We note that every regular D_{2m}-cover $\widetilde{\pi}\colon Y \to \mathbb{P}^1_x$ can be presented algebraically
$$y^{2m} - p(x)y^m + q(x)^m = 0 \tag{4.28}$$
for some $p, q \in \mathbb{C}[x]$. One of the central points of the discussion above is that when Y is associated to a dihedral form, the polynomials p and q can be chosen to have small degree.

4. *The divisorial equation.* Finally, for a more geometric perspective on the space $\Omega D_{m,n}$, we remark that whenever C is defined by $z^2 - pz + q^m = 0$ with $(p, q) \in P_{m,n}$, the divisor of z satisfies
$$(z) = m\left(\sum_1^k Q_i\right) - nP + (n - mk)P'$$
for some $Q_1, \ldots, Q_k \in C$. Here $k = \deg(q) \leq \lfloor 2n/m \rfloor$. Thus we obtain a close relationship between dihedral forms and the subvarieties of the moduli space $\mathcal{M}_{2,1}$ defined by:
$$\mathcal{D}_{m,n,k} = \left\{ (C,P) : \begin{array}{c} \text{there exist } Q_i \in C \text{ such that} \\ [nP + (mk-n)P'] = m[\sum_1^k Q_i] \end{array} \right\}.$$
Here $[D] \in \operatorname{Pic}(C)$ denotes the linear equivalence class of a divisor. Since $\dim \operatorname{Pic}(C) = 2$, we have $\mathcal{D}_{m,n,k} = \mathcal{M}_{2,1}$ for $k \geq 2$, while for $k < 2$ one expects $\mathcal{D}_{m,n,k}$ to have codimension $2 - k$, consistent with our calculation of $\dim \Omega D_{m,n}$.

   We have chosen to take a more algebraic approach to $\Omega D_{m,n}$ to avoid technical issues, such as the reducibility of $\mathcal{D}_{m,n,k}$.

5 The variety generated by a quadrilateral

In this section we prove Theorems 1.1 and 1.2: we show that the 6 values of (m, n) in Table 1 give new $\mathrm{SL}_2(\mathbb{R})$–invariant subvarieties
$$\overline{\Omega D}_{m,n} \subset \Omega\mathcal{M}_g,$$
and that these varieties are generated by quadrilaterals. We also explain how the table was derived, and describe properties of particular examples.

Statement of results. Let (m, n) be a dihedral type. By the character calculation in §3, we have $\dim \Omega D_{m,n} \leq 4$ (see Corollary 3.2). The main results of this section concern the case of equality. We will show:

Theorem 5.1 *Suppose* $\dim(\Omega D_{m,n}) = 4$*. Then the closure of* $\Omega D_{m,n}$ *in its stratum locally coincides, in period coordinates, with a finite union of subspaces of the form*

$$\bigcup_1^s D(X_i, \omega_i, T_i) \subset H^1(X, Z(\omega)).$$

(Here the dihedral triples on the left also have type (m, n).)

Corollary 5.2 *In this case* $\overline{\Omega D}_{m,n}$ *is an* $\mathrm{SL}_2(\mathbb{R})$*–invariant subvariety of* $\Omega\mathcal{M}_g$*.*

Proof. Any closed set in $\Omega\mathcal{M}_g$ that is locally defined by real linear equations is $\mathrm{SL}_2(\mathbb{R})$ invariant (see [MMW, §5]). ■

Theorem 5.3 *When* $\dim \Omega D_{m,n} = 4$*, the closure of the dihedral locus is generated by cyclic forms: we have*

$$\overline{\Omega D}_{m,n} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot \Omega Z_a} = \Omega G_a,$$

where $a = (a_1, a_2, a_3, a_4)$ *and* (m, n) *are related by equation (1.2).*

In fact ΩG_a can be generated by a single quadrilateral (see the closing remarks of this section).

The proof of Theorem 5.1 is self–contained and similar to the proof of linearity of the space of cyclic forms (Theorem 2.5), while the proof of Theorem 5.3 depends on the analysis of $\mathrm{SL}_2(\mathbb{R})$ orbits in [EMM].

The six exceptional quadrilaterals. We can now explain the derivation of Table 1. The main point is that Theorem 4.1 gives:

$$n + \gcd(m, n) \le m \le 2n \quad \Longrightarrow \quad \dim \Omega D_{m,n} = 4. \tag{5.1}$$

In particular Theorem 4.1 implies:

We have $\dim \Omega D_{m,n} = 4$ *for all six* (m, n) *listed in Table 1.*

There are only two other dihedral types satisfying the inequalities in (5.1): namely $(m, n) = (4, 3)$ and $(6, 5)$. But these correspond to previously known invariant varieties: they arise from the strata of quadratic differentials $Q\mathcal{M}_{1,3}(3, -1^3)$ and $Q\mathcal{M}_{2,1}(5, -1)$ respectively by a double covering construction. The other six are genuinely new. In fact we will show:

Theorem 5.4 *For the six values of* (m, n) *in Table 1, the locus*

$$\overline{\Omega D}_{m,n} \subset \Omega \mathcal{M}_g$$

is a primitive invariant variety that does not arise from a stratum of holomorphic 1-forms or quadratic differentials.

In this context *primitive* means a generic form in $\overline{\Omega D}_{m,n}$ is not the pullback of a form on a Riemann surface of lower genus.

Proof of Theorems 1.1 and 1.2. By Theorem 5.3 we have $\Omega G_a = \overline{\Omega D}_{m,n}$ for the six cases in Table 1; thus the desired statements follow from Theorem 5.4. ■

It remains to prove Theorems 5.1, 5.3 and 5.4.

Proof of Theorem 5.1. The argument follows the same lines as the proof of Theorem 2.5 for cyclic forms. Let $V = \Omega D_{m,n}$. For each dihedral form $(X, \omega) \in V$, choose T such that (X, ω, T) is a dihedral triple, and set $S(X, \omega) = D(X, \omega, T) = \mathrm{Ker}(T^* - \tau_m I)$. Then $\dim S(X, \omega) = \dim V = 4$ by Theorem 3.1, so the conclusion follows from Theorem 2.4. ■

Proof of Theorem 5.3. We have $\Omega G_a = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot \Omega Z_a}$ by definition, and $\Omega G_a \subset \overline{\Omega D}_{m,n}$ by Theorem 4.2. The latter variety is irreducible and 4–dimensional, so to show equality holds it suffices to show that ΩG_a too is an analytic variety with $\dim \Omega G_a \geq 4$.

By [EMM], the orbit closure of each point in ΩZ_a is equal to one of countably many analytic subvarieties in its stratum, each locally defined by real linear equations in period coordinates. Since ΩZ_a is irreducible, it is contained in exactly one of these varieties, and hence ΩG_a itself is $\mathrm{SL}_2(\mathbb{R})$–invariant and defined over $\mathbb{R}$. On the other hand, by Theorem 2.5, the variety ΩZ_a locally contains an open subset of a 2–dimensional subspace of the form $E = E(X, \omega, r) = \mathrm{Ker}(r^* - \zeta_m^{-1})$. Since ΩG_a is locally defined by real linear equations, and $\dim(E + \overline{E}) = 4$, we have $\dim \Omega G_a \geq 4$ and the proof is complete. ■

Proof of Theorem 5.4. Let (m, n) be one of the six dihedral types given in Table 1. Then $\dim \Omega D_{m,n} = 4$ by Theorem 4.1, so by Theorem 5.1 its closure is indeed a 4–dimensional invariant subvariety of $\Omega \mathcal{M}_g$. It remains to show this locus is (A) primitive, and (B) it does not arise from a stratum of holomorphic 1–forms or quadratic differentials.

(A). We have six values of (m,n) to consider. For each, consider a generic dihedral triple (X,ω,T) of type (m,n). We must show there does not exist a form (Z,η) of lower genus and a surjective holomorphic map $f : X \to Z$ such that $\omega = f^*(\eta)$.

If this is the case, then $[\omega] \in H^1(X)$ lies in a copy of the $2g(Z)$–dimensional subspace $H^1(Z)$ defined over $\mathbb{Q}$. Note that $[\omega]$ itself is a generic point in the 4–dimensional subspace $\operatorname{Ker}(T^* - \tau_m)$, which is defined over $K = \mathbb{Q}(\tau_m)$. Thus

$$2g(Z) = \dim H^1(Z) \geq 4\deg(K/\mathbb{Q}).$$

In particular, when $m = 5, 8$ or 10, K is quadratic over $\mathbb{Q}$, so $g(Z) \geq 4$. But $g(X) \leq 6$, so there is no surjective map $f : X \to Z$ and hence (X,ω) is primitive.

In the remaining two cases, $(m,n) = (6,3)$ and $(6,4)$ the only possibility is $g(Z) = 2$. We will see below that the $(6,3)$ case is the gothic locus, so its primitivity was already proved in [MMW, Lemma 6.2].

The final case, $(m,n) = (6,4)$, corresponds to $a = (1,1,2,8)$ and can be completed as follows. Consider a generic cyclic form (X,ω) of type a. It suffices to show that (X,ω) is primitive. If not, it is pulled back from a form (Z,η) of genus two via a map $f : X \to Z$. We cannot have $\deg(f) = 3$, since this would force f to be unramified and hence we would have $|Z(\omega)| \geq 3$, contrary to the fact that ω has just two zeros. Hence $\deg(f) = 2$, and therefore we have an involution $J \in \operatorname{Aut}(X)$ satisfying $J^*\omega = \omega$. Since $r^*\omega = \zeta_6\omega$, J is not in the cyclic group $\langle r\rangle \subset \operatorname{Aut}(X)$.

If the commutator $s = [J,r] \in \operatorname{Aut}(X)$ is trivial, then J descends to give a nontrivial symmetry of the quotient space $(X,|\omega|)/r$, a sphere with four cone points of type $(1,1,2,8)/6$. But generically this cone manifold has trivial isometry group, so no such J exists and (X,ω) is primitive.

Finally, if s is nontrivial, then $[s,r]$ is trivial since both s and r fix $Z(\omega)$ pointwise, and the same reasoning applies with J replaced by s. ∎

(B). In the six cases of interest, the inclusion of $\Omega D_{m,n}$ into its ambient stratum is proper because $\dim \Omega_g(v^u) \geq 2g > 4$ for all strata listed in Table 1. Thus ΩG_a does not arise from a stratum of holomorphic 1–forms.

Similarly, if $\Omega D_{m,n}$ comes from a stratum of quadratic differentials, it must be defined over $\mathbb{Q}$ and we only have the cases $(6,3)$ and $(6,4)$ to consider. Since a generic form (X,ω) in either locus is primitive, the only quadratic differential it covers is $(A,q) = (X,\omega^2)/J$, where $J = r^{m/2} = r^3$. The form (A,q) lies in the stratum $Q\mathcal{M}_{1,3}(-1^3,1^3)$ when $n = 3$ and

$Q\mathcal{M}_{2,2}(-1^2, 6)$ when $n = 4$. But these strata have dimensions 6 and 5 respectively, so they cannot account for the 4–dimensional variety $\Omega D_{m,n}$. ■

Remark. To check the dimension calculation in part (B), note that there are $g(A)$ conditions for a divisor of degree $4g(A)-4$ to come from a quadratic differential on A. More generally, $\dim Q\mathcal{M}_{g,n}(-1^n, p_1, \ldots, p_s) = 2g - 2 + n + s$; see [KZ, eq. (2)], [V1, eq. (0.9)]).

Particular examples. We now turn to particular cases of the general discussion above.

1. The variety $\Omega G_{1,1,1,9} \subset \Omega\mathcal{M}_4(2^3)$ is the *gothic locus*, denoted ΩG in [MMW]. To see this, note that each form $(X, \omega) \in \Omega G_{1,1,1,9}$ comes equipped with an involution $J = r^3 \in \mathrm{Aut}(X)$ and a degree three map $p : X \to B = Y/\langle f, r^2\rangle$ which satisfy the conditions for (X, ω) to be a gothic form, as defined in [MMW, §1]. Thus $\Omega G_{1,1,1,9}$ is contained in ΩG, and equality holds because both varieties are closed, irreducible and of dimension four.

2. The forms (X, ω) in $\Omega G_{1,1,2,8} \subset \Omega\mathcal{M}_4(3^2)$ have the interesting property that their zero set $Z(\omega) = \{P_1, P_2\}$ gives a divisor $[P_1 - P_2]$ of order 3 in $\mathrm{Jac}(X)$. Indeed, $3(P_1 - P_2) = (\phi)$, where $\phi : X \to Y/\langle r^2, f\rangle \cong \mathbb{P}^1$ is the natural quotient map.

3. Although it is not covered by Theorem 4.1, when $a = (1, 2, 2, 3)$ and $(m, n) = (4, 5)$ we also have $\dim \Omega D_{m,n} = 4$, so Theorems 5.1 and 5.3 apply. Indeed, for these values of a and (m, n) we have
$$\Omega G_a = \overline{\Omega D}_{m,n} = \Omega\mathcal{M}_2(2).$$
Given a form (X, ω) of genus two with a unique zero P, the desired degree four dihedral map $\pi : X \to \mathbb{P}^1$ can be obtained from a degree two hyperelliptic map $f : X \to \mathbb{P}^1$ by setting $\pi(z) = (f(z) - f(P))^2$.

   In this sense, the loci ΩG_a are generalizations of the stratum $\Omega\mathcal{M}_2(2)$.

4. The branch points $(b_1, \ldots, b_7)$ of the dihedral maps associated to forms in $\Omega D_{m,n}$ give an interesting hypersurface $V_{m,n}$ in $\mathcal{M}_{0,7}$. For example, the proof of Theorem 4.4 shows that $V_{6,3}$ consists of those points $(b_1, \ldots, b_6)$ that arise as the zeros of a sum of sextic polynomials of the form
$$c(x) = a(x)^2 + b(x)^6,$$
together with the point $b_7 = \infty$.

5. The Teichmüller curve associated to the $(3,5,7)$ triangle lies in $G_{1,1,1,7}$. To see this, note that the cyclic form $\omega = dx/y$ on the curve X defined by $y^5 = (x^3-1)^4$ lies in ΩZ_a. This form is equal to the unfolding of the $(3,5,7)$-triangle, as can be seen by considering the cyclic, degree 15 map $x^3 \colon X \to \mathbb{P}^1$. For a proof that this form generates a Teichmüller curve, and its connection to the E_8–diagram, see [Vo] and [Lei].

General properties. We conclude with some remarks that apply to all six loci ΩG_a given in Table 1. For any a from this list:

1. The locus ΩG_a lies in a single stratum $\Omega\mathcal{M}_g(v^u)$. This is easy to see when $|Z(\omega)| = 1$ and when $r^{m/2}$ fixes $Z(\omega)$ pointwise. The only remaining case is $a = (1,1,2,8)$, which can be handled using the torsion property mentioned above.

2. The locus ΩG_a has rank two, by Theorems 3.1 and 5.1.

   Here the *rank* of an irreducible invariant variety $V \subset \Omega\mathcal{M}_g$ is one–half the dimension of its projection to $H^1(X)$ in period coordinates (see [Wr2]). The previously known examples of higher rank V all come from strata; this is not the case for ΩG_a, by Theorem 5.4. It is known that for each fixed g, there exist only finitely many invariant subvarieties of $\Omega\mathcal{M}_g$ of rank two or more [EFW, Theorem 1.3].

3. There exists a planar quadrilateral of type a whose associated form generates ΩG_a. Indeed, by [EMM], every form in ΩZ_a whose $\mathrm{SL}_2(\mathbb{R})$ orbit is not dense in ΩG_a lies in one of countably many proper subvarieties of ΩZ_a. A generic quadrilateral will avoid these.

4. By passing to the closure of ΩG_a in a suitable compactification of $\Omega\mathcal{M}_g$ (e.g. the one appearing in [Mc5, §4]), one obtains further interesting examples of $\mathrm{SL}_2(\mathbb{R})$–invariant varieties of rank one and dimension three.

6 Totally geodesic surfaces

In this section we prove Theorem 1.3 on the existence of a totally geodesic surface in $\mathcal{M}_{1,3}$, $\mathcal{M}_{1,4}$ and $\mathcal{M}_{2,1}$, and make remarks on each of the three examples.

Quadratic differentials. Let (m,n) be a dihedral type with m even. Then for every dihedral triple (X,ω,T), we have a natural involution $J \in \mathrm{Aut}(X)$ satisfying

$$J^*(\omega) = -\omega.$$

This involution is defined using the action of D_{2m} on Y constructed in §3, by observing that the map $r^{m/2} \in \mathrm{Aut}(Y)$ commutes with f, and so it descends to an involution J on $X = Y/f$.

Letting $r = g(X/J)$ and $s = |\mathrm{Fix}(J) - Z(\omega)|$, we then have a natural subvariety

$$QF_{m,n} \subset Q\mathcal{M}_{r,s}$$

consisting of all quadratic differential of the form

$$(A, q) = (X, \omega)/J$$

arising from dihedral triples of type (m, n). The original form can be recovered from (A, q) by taking the Riemann surface of $\sqrt{q}$, so

$$\dim QF_{m,n} = \dim \Omega D_{m,n} = 4.$$

Let $F_{m,n}$ denote the closure of its projection to $\mathcal{M}_{r,s}$. The following is a restatement of Theorem 1.3.

Theorem 6.1 *Let (m, n) be a dihedral type with $m = 2n \geq 6$. Then $F_{m,n} \subset \mathcal{M}_{r,s}$ is a primitive, totally geodesic surface.*

Proof. Since $n = 3, 4$ or 5, there are only three cases to consider, and they all appear in Table 1. In all cases the locus $\Omega G_a = \overline{\Omega D}_{m,n} \subset \Omega\mathcal{M}_g$ is a primitive, 4–dimensional, $\mathrm{SL}_2(\mathbb{R})$ invariant variety, so the same is true for the closure of $QF_{m,n}$.

Since the monodromy about b_{p+1} for a dihedral form of type (m, n) (with $m = 2n$ and $p = 6$) is $[r^n] = [J]$, the quotient $A = X/J$ and the set of poles P of (A, q) do not depend on the location of the special critical value b_{p+1}. On the other hand, as b_{p+1} varies while keeping $(b_1, \ldots, b_p)$ fixed, the zeros of q move, since they are the fiber over b_{p+1}; so we obtain a two–dimensional family of quadratic differentials in $Q(A, P)$. The corresponding double covers give a two–dimensional family of 1–forms $(X, \omega) \in \Omega\mathcal{M}_g$. Since the monodromy of the map $X \to \mathbb{P}^1$ remains constant as b_{p+1} varies, and the special fiber X^* over b_{p+1} supports the zeros of $\omega = \sqrt{q}$, this two–dimensional family lies in $\Omega D_{m,n}$ by the definition of a dihedral form. Thus the fibers of the composed map $\Omega D_{m,n} \to QF_{m,n} \to F_{m,n}$ are at least two–dimensional, and hence $\dim F_{m,n} \leq 2$. On the other hand, $F_{m,n}$ is the image of an invariant variety of quadratic differentials of dimension 4, so $\dim(F_{m,n}) \geq 2$. We conclude that $F_{m,n}$ is a complex surface and that it contains a pencil of complex geodesics through each point. Consequently $F_{m,n}$ is totally geodesic, and it is primitive because ΩG_a is primitive. ∎

For a more detailed treatment of the case $(6,3)$, see [MMW, §5].

We conclude with some remarks on each of these three surfaces.

Plane cubics and $\boldsymbol{F_{6,3}}$. As we have seen at the end of §5, the closure of $\Omega D_{6,3}$ gives the gothic locus $\Omega G \subset \Omega\mathcal{M}_4$. Consequently $F_{6,3}$ coincides with the *flex locus* $F \subset \mathcal{M}_{1,3}$. A quite different description of this locus, in terms of classical algebraic geometry and plane cubics, is given in [MMW, Theorem 3.1].

Space curves and $\boldsymbol{F_{8,4}}$. Similarly, the surface $F_{8,4} \subset \mathcal{M}_{1,4}$ can be described in terms of pencils of quadrics in $\mathbb{P}^3$. The base locus of such a pencil is a quartic curve $A \cong X/J$ of genus one, and the dihedral map $\pi : X \to \mathbb{P}^1$ factors through a degree four map $p_L : A \to \mathbb{P}^1$ obtained by projection from a suitable line $L \subset \mathbb{P}^3$.

Real multiplication and $\boldsymbol{F_{10,5}}$. Finally, the surface $F_{10,5} \subset \mathcal{M}_{2,1}$ is closely related to a remarkable construction that associates a point $P \in X$ to each eigenform (X,ω) for real multiplication by the ring of integers in $\mathbb{Q}(\sqrt{5})$ on the Jacobian of a Riemann surface of genus 2. The surface $F_{10,5}$ consists of the pairs (X,P) that arise in this way. For more details, see [KM] and [Ap].

7 Teichmüller curves

In this section we prove Theorem 1.4: we show the 3–fold $G_a \subset \mathcal{M}_4$ is abundantly populated by primitive Teichmüller curves for $a = (1,1,1,9)$ and $a = (1,1,2,8)$. Explicit examples, coming from billiards, are presented in §8. The proof is based on eigenforms for real multiplication, as in the construction of Weierstrass curves in [Mc1].

Real multiplication. Let $K \subset \mathbb{R}$ be a real quadratic field, with Galois involution $x \mapsto x'$, and let A be a polarized Abelian variety with $\dim(A) = 2$.

We say A admits *real multiplication* by K if we have an inclusion

$$K \subset \mathrm{End}(A) \otimes \mathbb{Q}$$

whose image consists of self–adjoint operators. We say $\omega \in \Omega(A)$ is an *eigenform* for real multiplication by K if $K \cdot \omega \subset \mathbb{C} \cdot \omega$. This is equivalent to the requirement that

$$R^*(\omega) = \lambda\omega \quad \text{and} \quad \mathbb{Q}(\lambda) = K,$$

for some self–adjoint endomorphism $R \in \mathrm{End}(A)$.

Let $\mathcal{O}_D \cong \mathbb{Z}[(D+\sqrt{D})/2]$ be the real quadratic order of discriminant D. When we wish to attend to the integral structure, we say A admits real multiplication by $\mathcal{O}_D$ if

$$K \cap \mathrm{End}(A) \cong \mathcal{O}_D\,.$$

In this case $K = \mathbb{Q}(\sqrt{D})$.

Density. Eigenvectors for real multiplication are ubiquitous, even for a fixed field K. We will use the following fact to prove density of Teichmüller curves.

Proposition 7.1 *Let $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ be the standard symplectic form on $\mathbb{Q}^4$, and let λ be a real quadratic number. Let $E_\lambda \subset \mathbb{C}^4$ be the union of the λ–eigenspaces of all self–adjoint operators $R \in M_4(\mathbb{Q})$. Then E_λ is dense in $\mathbb{C}^4$.*

(Self–adjoint means $JR = R^t J$.)

Proof. The matrix $R = \left(\begin{smallmatrix} A & 0 \\ 0 & A^t \end{smallmatrix}\right)$ is self–adjoint for any $A \in GL_2(\mathbb{Q})$. It is easy to choose A so that the eigenvalues of R are λ and λ^{-1}. Then $S = \mathrm{Ker}(R - \lambda I)$ is a 2–dimensional symplectic subspace of $\mathbb{R}^4$. Now recall that $\mathrm{Sp}_4(\mathbb{Q})$ is dense in $\mathrm{Sp}_4(\mathbb{R})$, and the latter group acts transitively on the set of 2–dimensional symplectic subspaces. Thus given any $z \in \mathbb{C}^4$, we can perturb it slightly so that $\mathrm{Re}(z), \mathrm{Im}(z) \in gS$ for some $g \in \mathrm{Sp}_4(\mathbb{Q})$. But gS is the λ–eigenspace for $gRg^{-1} \in M_4(\mathbb{Q})$, so $z \in E_\lambda$. Therefore E_λ is dense in $\mathbb{C}^4$. ∎

Note that the density of $\mathrm{Sp}_4(\mathbb{Q})$ in $\mathrm{Sp}_4(\mathbb{R})$ is a special case of weak approximation for algebraic groups.

The quotient torus of a 1-form. Next we relate 1–forms and Abelian varieties. Let us say a subspace $S \subset H^1(X)$ is *compatible* with the Hodge structure if

$$S = S^{1,0} \oplus S^{0,1},$$

where $S^{i,j} = S \cap H^{i,j}(X)$.

Any form $(X, \omega) \in \Omega\mathcal{M}_g$ determines a canonical *quotient torus*

$$p : \mathrm{Jac}(X) \to A_\omega,$$

characterized by the property that, under pullback on cohomology,

$$H^1(A_\omega) \subset H^1(X)$$

is the smallest subspace containing $[\omega]$, defined over $\mathbb{Q}$ and compatible with the Hodge structure on $H^1(X)$; and

$$H^1(A_\omega, \mathbb{Z}) = H^1(A_\omega) \cap H^1(X, \mathbb{Z}).$$

In particular, we have $\omega \in \Omega(A_\omega) \subset \Omega(X)$. The torus A_ω inherits a polarization from $\mathrm{Jac}(X)$ making it into an Abelian variety.

The dihedral case. Now suppose $a = (a_1, a_2, a_3, a_4)$ and (m, n) are related by (1.2) as usual, and that $\dim \Omega D_{m,n} = 4$. Then $\Omega G_a = \overline{\Omega D}_{m,n}$ and ΩG_a is locally linear in period coordinates (§5).

Let $\phi(m)$ denote the Euler ϕ–function and recall that $\tau_m = \zeta_m + \zeta_m^{-1}$.

Proposition 7.2 *For any $(X, \omega) \in \Omega G_a$, we have $\dim(A_\omega) \le \phi(m)$.*

Proof. Since $\dim(A_\omega)$ achieves its maximum at a very generic point, it suffices to prove this when $(X, \omega) \in \Omega D_{m,n}$. In this case we have an algebraic correspondence $T \subset X \times X$ such that $[\omega] \in S_0 = \mathrm{Ker}(T^* - \tau_m I) \subset H^1(X)$. Let $S \subset H^1(X)$ be the smallest subspace defined over $\mathbb{Q}$ and containing S_0. Since $\deg(\tau_m/\mathbb{Q}) = \phi(m)/2$ and $\dim(S_0) \le 4$ by Theorem 3.1, we have $\dim(S) \le 2\phi(m)$. Since T is compatible with the Hodge structure on $H^1(X)$, so is S. Therefore $H^1(A_\omega) \subset S$, which implies that $\dim(A_\omega) = (1/2) \dim H^1(A_\omega) \le \phi(m)$ as desired. ∎

Dihedral eigenforms. For the remainder of this section, let us also assume that $m = 3, 4$ or 6. Then $\phi(m) = 2$, $\tau_m = 0, 1$ or -1, and we have

$$\dim A_\omega \le 2$$

for all $(X, \omega) \in \Omega G_a$, by the result above.

For each non-square discriminant $D > 0$, let

$$\Omega G_{a,D} = \left\{ (X, \omega) \in \Omega G_a \ : \ \begin{array}{c} \dim(A_\omega) = 2 \text{ and } \omega \text{ is an eigenform} \\ \text{for real multiplication by } \mathcal{O}_D \text{ on } A_\omega \end{array} \right\}.$$

The projection of $\Omega G_{a,D}$ to $\mathcal{M}_g$ will be denoted by $G_{a,D}$.

Proposition 7.3 *The locus $\Omega G_{a,D}$ is a closed algebraic subvariety of $\Omega \mathcal{M}_g$, and each of its components has dimension at least two.*

Proof. Let $\Omega G'_a$ denote the smooth points of ΩG_a. By Theorem 5.1, ΩG_a locally has the form $\bigcup_1^s D(X_i, \omega_i, T_i)$ in period coordinates, so at a non-smooth point $[\omega]$ belongs the intersection of two 4–dimensional subspaces, and hence $\dim(A_\omega) = 1$. Thus $\Omega G_{a,D}$ is a subset of $\Omega G'_a$, and at these points $s = 1$. Moreover $T_1 : H^1(X) \to H^1(X)$ is represented by an endomorphism $T \in \mathrm{Jac}(X)$, and

$$H^1(A_\omega) = \mathrm{Ker}(T^* - \tau_m I) \subset H^1(X),$$

provided $\dim(A_\omega) = 2$. Now recall that every eigenvalue of T^* is an integer between -2 and 2. Thus, if we define

$$\alpha(X, \omega) = \mathrm{Jac}(X) \Bigg/ \mathrm{Ker}\left(\prod_{|k| \leq 2,\ k \neq \tau_m} (T^* - kI) \right),$$

we obtain an analytic map

$$\alpha : \Omega G'_a \to \Omega \mathcal{A}_2$$

such that $\alpha(X, \omega) = (A_\omega, \omega)$ whenever $\dim(A_\omega) = 2$. Here $\mathcal{A}_2$ is the moduli space of two–dimensional Abelian varieties with the same polarization as A_ω for a generic form in ΩG_a.

Using the algebraic parameterization of the space of dihedral triples given by Theorem 4.3, it is easy to see that α agrees with an algebraic function on the Zariski open set $\Omega D_{m,n} \cap \Omega G'_a$. Hence α itself is algebraic. On the other hand, $\Omega G_{a,D}$ is simply the preimage under α of the locus of eigenforms for real multiplication by $\mathcal{O}_D$. The latter is an algebraic variety of codimension two in $\Omega \mathcal{A}_2$ that can be described in terms of Hilbert modular surfaces (cf. [vG, Ch. 9]). Thus $\Omega G_{a,D}$ is also algebraic variety, and each of its irreducible components has dimension ≥ 2 since $\dim \Omega G_a = 4$. To see that $\Omega G_{a,D}$ is closed in $\Omega \mathcal{M}_g$, use the fact that the ring of endomorphisms $\mathcal{O}_D \subset \mathrm{End}(A_\omega)$ is determined uniquely by the associated eigenform (A_ω, ω). ■

Theorem 7.4 *The locus $\Omega G_{a,D}$ is a finite union of closed $\mathrm{GL}_2^+(\mathbb{R})$ orbits.*

Proof. Let $\mathcal{O}_D = \mathbb{Z}[\lambda] \subset \mathbb{R}$. Given $(X, \omega) \in \Omega G_{a,D}$ we can find, by the definitions and the discussion above, an operator T on $H^1(X, Z(\omega))$ and an $R \in \mathrm{End}(A_\omega)$ such that

$$T^*[\omega] = \tau_m[\omega] \quad \text{and} \quad R^*\omega = \lambda\omega.$$

These conditions give a 2–dimensional space $S(X,\omega) \subset H^1(X, Z(\omega))$, defined over $\mathbb{Q}(\lambda)$, containing $[\omega]$. Since $\dim \Omega G_{a,D} \geq 2$ by Proposition 7.3, Theorem 2.4 implies that $\Omega G_{a,D}$ is two–dimensional and locally defined by real linear equations in period coordinates. Hence $\Omega G_{a,D}$ is $\mathrm{GL}_2^+(\mathbb{R})$–invariant (cf. [MMW, §5])). Each orbit gives an irreducible component of $\Omega G_{a,D}$, so their total number is finite. ■

Corollary 7.5 *The variety $G_{a,D}$ is a finite union of Teichmüller curves.*

Corollary 7.6 *The stabilizer $\mathrm{SL}(X,\omega)$ of any form in $\Omega G_{a,D}$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.*

Proposition 7.7 *Eigenforms for real multiplication are dense in ΩG_a. In fact for each real quadratic field K, the countable union of $\mathrm{SL}_2(\mathbb{R})$–orbits*

$$Z_K = \bigcup_{D\,:\,\mathbb{Q}(\sqrt{D})=K} \Omega G_{a,D}$$

is dense in ΩG_a.

Proof. It suffices to prove that the closure of Z_K contains those forms

$$(X,\omega) \in \Omega D_{m,n} \subset \Omega G_a$$

with $\dim H^1(A_\omega) = 4$, since such forms are dense in ΩG_a.

Let (X,ω,T) be a dihedral triple associated to such a form. Then the natural projection from relative to absolute cohomology gives an isomorphism

$$D(X,\omega,T) \cong H^1(A_\omega) \subset H^1(X) \tag{7.1}$$

by Theorem 3.1, and $\Omega D_{m,n}$ meets the space at the left in an open neighborhood of $[\omega]$ in period coordinates.

Let $K = \mathbb{Q}(\lambda)$, and note that the symplectic space $H^1(A,\mathbb{Q})$ is isomorphic to $(\mathbb{Q}^4, J)$. By Proposition 7.1, there exists a self–adjoint endomorphism R of $H^1(A,\mathbb{Q})$, and a cohomology class $[\eta]$ near $[\omega]$, such that $R([\eta]) = \lambda[\eta]$. Using the isomorphism (7.1), we can lift this class to a 1–form (W,η) near (X,ω) in $\Omega D_{m,n}$. Clearly R preserves the Hodge structure on

$$S = \mathbb{C}\cdot[\eta] \oplus \mathbb{C}\cdot[\overline{\eta}] \subset H^1(A_\eta) \cong H^1(A_\omega).$$

But R acts by the scalar λ' on $S^\perp = S'$, so it preserves the Hodge structure on $H^1(A_\eta)$. Hence R makes η into an eigenform for real multiplication by K on A_η. Since (W,η) can be chosen arbitrarily close to (X,ω), density follows. ■

Corollary 7.8 *The Teichmüller curves $G_{a,D}$ are dense in G_a.*

Proposition 7.9 *For $a = (1,1,1,9)$ or $(1,1,2,8)$, every component of $G_{a,D}$ is a primitive Teichmüller curve.*

(In these cases $m = 6$.)

Proof. Recall that a generic form in ΩG_a is primitive by Theorem 1.1. Thus an imprimitive form must satisfy an additional rational condition in relative period coordinates. But the only subspace of $H^1(A)$ defined over $\mathbb{Q}$ and containing the class of an eigenform for real multiplication is $H^1(A)$ itself. ■

Proof of Theorem 1.4. Combine Corollary 7.8 and Proposition 7.9. ■

Remark. When $a = (1,2,2,3)$, the locus $G_{a,D}$ coincides with the Weierstrass curve $W_D \subset \mathcal{M}_2$ defined in [Mc1]. For $a = (1,1,1,5)$ and $a = (1,2,2,7)$, we recover the Teichmüller curves in $\mathcal{M}_3$ and $\mathcal{M}_4$ constructed using Prym varieties in [Mc3].

8 Billiards

In this section we prove Theorem 1.5 on optimal billiards. That is, we will show that suitable (a,b), the quadrilaterals shown in Figure 2 give cyclic forms that generate Teichmüller curves.

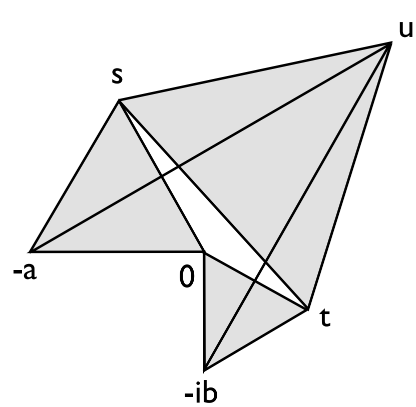

Figure 3. An unfolded quadrilateral of type $(1,1,1,9)$.

Unfolding. For concreteness let us treat the case of quadrilaterals of type $(1,1,1,9)$. Let Σ_4 be an oriented topological surface of genus 4. We begin by defining, for each (s,t) in a suitable open set $U \subset \mathbb{C}^2$, a holomorphic 1-form

$$\Phi(s,t) = (X,\omega) \in \Omega Z_{1119},$$

and a linear map

$$\phi : \mathbb{C}^2 \to H^1(\Sigma_4),$$

defined over $\mathbb{Q}(\zeta_3)$, such that the cohomology class of ω is given by

$$[\omega] = \phi(s,t) \in H^1(X) \cong H^1(\Sigma_4) \tag{8.1}$$

with respect to a compatible marking of X by Σ_4.

The form $\phi(s,t) = (X,\omega)$ will be constructed geometrically. First, consider the case where we have a pair of real numbers $a, b > 0$ such that $(s,t) = (\zeta_3 a, \zeta_{12}^{-1} b)$. Construct the triangle in the complex plane with vertices $(0,t,s)$, and attach the three equilateral triangles to its sides, by adding new vertices at $-a$, $-ib$ and u. The result is a 6-sided polygon $P(s,t) \subset \mathbb{C}$ (see Figure 3). Now note that the form dz^6 on $\mathbb{C}$ is invariant under $z \mapsto \zeta_6 z$. Thus, by gluing together the free edges of each equilateral triangle using a rotation by 60°, we obtain a copy of the projective line equipped with a meromorphic section ξ of $K^6_{\mathbb{P}^1}$:

$$(\mathbb{P}^1, \xi) = (P(s,t), dz^6)/\sim .$$

There is then a natural $\mathbb{Z}/6$ branched covering space $\pi : X \to \mathbb{P}^1$, itself equipped with a holomorphic 1-form $\omega \in \Omega(X)$, such that $\pi^*(\xi) = \omega^6$, and we let

$$\Phi(s,t) = (X,\omega).$$

It is easy to see that the quadrilateral with vertices $(-a, 0, -ib, u)$ is the same as the quadrilateral $Q = Q_{1119}(a,b)$ in Figure 2, and the quotient $P(s,t)/\sim$ is its double. Thus (X,ω) is simply the 1-form that results by unfolding Q. In particular, we have $(X,\omega) \in \Omega Z_{1119}$.

Complexified quadrilaterals. Next observe that the construction of (X,ω) makes sense whenever the polygon $P(s,t)$ is embedded in $\mathbb{C}$. Moreover, the relative periods of ω are complex linear functions of (s,t). Thus we can find a small open neighborhood U of $\zeta_3\mathbb{R}_+ \times \zeta_{12}^{-1}\mathbb{R}+ \subset \mathbb{C}^2$ such that Φ extends to a holomorphic map

$$\Phi : U \to \Omega Z_{1119}.$$

Using the triangles shown in Figure 3, we obtain compatible triangulations of the corresponding surfaces X as (s,t) varies in U. The bundle of such surfaces over U is thus topologically trivial, and hence we can consistently mark these surfaces by Σ_4. Since 0, s and t are identified under gluing, the numbers $(s,t) \in \mathbb{C}^2$ are simply two of relative periods given by $\int_C \omega$ for suitable $C \in H_1(X, Z(\omega); \mathbb{Z})$. As can be seen from Figure 3, linear combinations of s and t over $\mathbb{Q}(\zeta_3)$ determine all the absolute periods of ω. Hence the map $\phi : U \to H^1(\Sigma_4)$ defined by equation (8.1) extends to a linear map on $\mathbb{C}^2$, defined over $\mathbb{Q}(\zeta_3)$.

The area form. Recall that $H^1(\Sigma_4)$ carries a natural Hermitian inner product of signature $(4,4)$, defined at the level of closed 1–forms by

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_{\Sigma_4} \alpha \wedge \overline{\beta}.$$

It is straightforward to compute the pullback of this form to $\mathbb{C}^2$ by ϕ. By linearity, it suffices to compute the corresponding quadratic form on the image of U.

Proposition 8.1 *Suppose $(X,\omega) = \Phi(s,t)$. Then we have*

$$\int_X |\omega|^2 = \frac{3}{2} \begin{pmatrix} s & t \end{pmatrix} M \begin{pmatrix} \overline{s} \\ \overline{t} \end{pmatrix},$$

where

$$M = \sqrt{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Proof. This is a straightforward calculation. The area of an equilateral triangle with unit sides is $\sqrt{3}/4$, and the area of the triangle with vertices $(0,s,t)$ is $\mathrm{Im}(s\overline{t})/2$. Therefore

$$\mathrm{area}(P(s,t)) = \frac{s\overline{t} - \overline{s}t}{4i} + \frac{\sqrt{3}}{4} \left(|s|^2 + |t|^2 + |s-t|^2 \right),$$

and we have $\int_X |\omega|^2 = 6 \,\mathrm{area}\, P(s,t)$ since $\pi : X \to \mathbb{P}^1$ has degree 6. ■

Eigenforms. Now let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, and let $L = K(\zeta_3)$. The group $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2)^2$, with generators $z \mapsto \overline{z}$

and $z \mapsto z'$, the latter chosen so that $(\zeta_3)' = \zeta_3$. Since ϕ is defined over $\mathbb{Q}(\zeta_3)$, it restricts to a map

$$\phi : L^2 \to H^1(\Sigma_g, L)$$

compatible with the action of $\mathrm{Gal}(L/\mathbb{Q}(\zeta_3))$. Let $[\alpha']$ denote the Galois conjugate of a class $[\alpha] \in H^1(X, L)$, and similarly for $[\overline{\alpha}']$.

Proposition 8.2 *Suppose* $(X, \omega) = \Phi(s, t)$, *and* $\int_X \omega \wedge \overline{\omega}' = 0$. *Then* (X, ω) *generates a primitive Teichmüller curve in* $\mathcal{M}_4$, *and its stabilizer* $\mathrm{SL}(X, \omega)$ *is a lattice in* $\mathrm{SL}_2(\mathbb{R})$.

A similar result appears in [Mc2, Cor. 9.10].

Proof. Let S be the span of ω and $\overline{\omega}$ in $H^1(X)$, and let S' be its Galois conjugate, spanned by ω' and $\overline{\omega}'$. We claim $S' \subset S^\perp$. Indeed, $\langle \omega, \omega' \rangle = 0$ by assumption, and $\langle \omega, \overline{\omega}' \rangle = 0$ since these forms lie in different eigenspaces of $r \in \mathrm{Aut}(X)$. Since $S \oplus S'$ is the smallest subspace of $H^1(X)$ containing $[\omega]$ and invariant under $\mathrm{Gal}(L/\mathbb{Q})$, in the notation of §7 we have $S \oplus S' = H^1(A_\omega)$ and hence $\dim(A_\omega) = 2$. Let R be the unique self–adjoint operator on $H^1(A_\omega)$ acting by scalar multiplication by $+\sqrt{d}$ on S and $-\sqrt{d}$ on S'. Then R is defined over $\mathbb{Q}$ and respects the Hodge structure on $H^1(A_\omega)$, so it determines an inclusion

$$K \subset \mathrm{End}(A_\omega).$$

This shows that $(X, \omega) \in \Omega G_{a,D}$ for some order $\mathcal{O}_D \subset K$; thus (X, ω) generates a primitive Teichmüller curve by Proposition 7.9, and hence its stabilizer is a lattice in $\mathrm{SL}_2(\mathbb{R})$. ∎

Corollary 8.3 *If, in addition,* $a = \zeta_3^{-1} s$ *and* $b = \zeta_{12} t$ *are both positive, then billiards in* $Q_{1119}(a, b)$ *has optimal dynamics.*

Proof. This is a well–known consequence of the fact that $\mathrm{SL}(X, \omega)$ is lattice; see e.g. [V2, Prop. 2.11], [MT, Thm. 5.10] and [D]. ∎

Proof of Theorem 1.5. Let $(a, b) = (1, \sqrt{3}y)$ where $y > 0$ is a quadratic irrational satisfying $y^2 + (3c+1)y + c = 0$ for some $c \in \mathbb{Q}$. Let $L = \mathbb{Q}(\zeta_3, y)$. Then the unfolding of the quadrilateral $Q_{1119}(a, b)$ is given by the form $(X, \omega) = \phi(s, t)$, where

$$(s, t) = (\zeta_3, \sqrt{3}\, \zeta_{12}^{-1}\, y) = (\zeta_3, (2 + \zeta_3^{-1})\, y) \in L^2.$$

We then compute, using the bilinear form from Proposition 8.1, that

$$\frac{i}{2}\int_X \omega\wedge\overline{\omega}' = \begin{pmatrix} s & t\end{pmatrix} M \begin{pmatrix}\overline{s}' & \overline{t}'\end{pmatrix}^t = 3\sqrt{3}(1+y+y'+3yy').$$

Since $yy'=c$ and $y+y'=-(3c+1)$, this gives $\int_X \omega\wedge\overline{\omega}'=0$, and the proof is completed by Proposition 8.2.

The corresponding result for $Q_{1128}(a,b)$ follows from a similar calculation, with $(s,t)=(\zeta_3 a,(1+\zeta_6^{-1})b)$ and

$$M=\sqrt{3}\begin{pmatrix} 2 & -1\\ -1 & 4/3\end{pmatrix}+\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}.$$

■

Remarks.

1. The forms (X,ω) that generate Teichmüller curves by Proposition 8.2 are actually eigenforms for *complex multiplication* by L on A_ω.

2. In fact, eigenforms for complex multiplication are dense in ΩZ_a when $a=(1,1,1,9)$ or $(1,1,2,8)$. To see this, note that if $\langle\omega,\omega'\rangle=0$, and $k=\int\omega\wedge\overline{\omega}$, then
$$\langle\omega+s\omega',\omega'+s'\omega\rangle=0$$
for all $s\in L$ satisfying $sk'+\overline{s}'k=0$. The set of such s includes k itself, and is invariant under multiplication by $\mathbb{Q}(\sqrt{-3d})$, so it is dense in $\mathbb{C}$. To complete the proof, use the irreducibility of ΩZ_a and its linearity in period coordinates (§2).

3. The fact that $\det(M)>0$ can be used to show that for all $(X,\omega)\in\Omega Z_{1119}$, we have an isogeny $\mathrm{Jac}(X)\to J(X)\times A$ where $\dim A=2$, A is independent of X, and $\omega\in\Omega(A)$. See the case $(n,d)=(3,6)$ of [Mc4, Thm 8.3].

4. Let $Q_{1218}(a,b)$ denote the variant of $Q_{1128}(a,b)$ where the sharpest angles are opposite one another. As before, a and b are the lengths of the edges incident to the obtuse vertex of Q. Then $Q_{1218}(1,y)$ is a lattice polygon whenever $y>0$ is an irrational satisfying $y^2+(2c+2)y+c=0$ for some $c\in\mathbb{Q}$. The calculation is very similar to the 1128 case.

5. Hooper's optimal triangle T [Ho], of type $(1, 4, 7)$, is closely related to the quadrilaterals of type $a = (1, 2, 2, 7)$, which arise in our discussion when $(m, n) = (6, 5)$. Indeed, a symmetric quadrilateral of type a can be cut along its diagonal to yield two copies of T. These quadrilaterals do not appear in our Table 1 because the locus ΩG_a is *not primitive*; in fact it is a Prym stratum in $\Omega M_4(6)$. The primitive Teichmüller curves associated to this Prym stratum, including the one generated by T, were first described in [Mc3].

6. There are infinitely many quadrilaterals of type $(1, 2, 2, 3)$ (with the angles in any cyclic order) that generate primitive Teichmüller curves $V \subset \mathcal{M}_2$. These cyclic forms give rise to orbifold points of order 2 on V; they are classified in [Mu].

   Similarly, the optimal quadrilaterals of types $(1, 1, 1, 9)$ and $(1, 1, 2, 8)$ give rise to orbifold points of order 3 on the associated Teichmüller curves $V \subset \mathcal{M}_4$.

   On the other hand, many Teichmüller curves (such as the Weierstrass curves of the form $W_{4D} \subset \mathcal{M}_2$) have no orbifold points [Mu], so they cannot be generated by cyclic forms.

A Triangles revisited

The Veech and Ward triangles, discovered in [V2] and [Wa], give two infinite series of triangles that generate Teichmüller curves.

In this section we show that these series of triangles fit into the same pattern as the exceptional quadrilaterals discussed in the body of this paper. That is, they arise naturally when one considers dihedral covers of $\mathbb{P}^1$ of type (m, n, p) with $p = 4$ instead of $p = 6$.

To make this connection precise, we will use the results of §3 to sketch a new proof of:

Theorem A.1 (Veech, Ward) *Let $a = (1, 1, q)$ or $(1, 2, 2q-1)$ with $q \geq 1$. Then the cyclic forms of this type generate a closed, 2–dimensional $\mathrm{SL}_2(\mathbb{R})$–invariant subvariety*

$$\Omega G_a = \mathrm{SL}_2(\mathbb{R}) \cdot \Omega Z_a \subset \Omega \mathcal{M}_g,$$

and the projection of ΩG_a to $\mathcal{M}_g$ is a Teichmüller curve.

As a corollary, the billiards in a triangle of type $(1,1,q)$ or $(1,2,q)$ has optimal dynamics.

Along the way we will obtain the first explicit algebraic formula, equation (A.6), for the Teichmüller curves generated by Ward triangles. The connection between dihedral groups and the Veech and Ward examples is also discussed, from a different point of view, in [Loch] and [BM]. A related construction, using the action of D_{2m} on an elliptic curve instead of on $\widehat{\mathbb{C}}$, appears in [Me].

Forms with zeros at infinity. To begin, we note that if Z is a compact Riemann surface defined by a polynomial equation of the form

$$P(x) = Q(y),$$

then the poles of x and y occur on the same set $Z^* \subset Z$. Moreover, if P and Q have distinct critical values in $\mathbb{C}$, then the divisor of the 1-form

$$\omega = \frac{dy}{P'(x)} = \frac{dx}{Q'(y)} \tag{A.1}$$

is a multiple of Z^*. If the critical values are not distinct, but the zeros of $P'(x)$ and $Q'(x)$ are simple, then a form with $(\omega) = v \cdot Z^*$ is given by

$$\omega = \frac{R(x)\,dy}{P'(x)}, \tag{A.2}$$

where zeros of $R(x)$ coincide with the critical points of $P(x)$ that map to critical values of $Q(y)$.

The case $\boldsymbol{p = 2}$. We now turn to dihedral covers. Let

$$C(z) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

so that $C(e^{i\theta}) = \cos\theta$. Recall that the Chebyshev polynomial $T_m(y)$ is uniquely determined by the condition that

$$T_m(C(z)) = C(z^m).$$

The simplest dihedral cover comes from the action of D_{2m} on $Y = \mathbb{P}^1$, given by

$$r(z) = \zeta_m z \quad \text{and} \quad f(z) = 1/z. \tag{A.3}$$

In this case $X = Y/f$ and Y/D_{2m} have genus zero, and the quotient maps $\xi : Y \to X$ and $\widetilde{\pi} : Y \to Y/D_{2m}$, in suitable coordinates, are given simply by $\xi(z) = C(z)$ and $\widetilde{\pi}(z) = C(z^m)$. Since we also have

$$\widetilde{\pi}(z) = T_m(C(z)) = T_m(\xi(z)),$$

the induced dihedral map on X is given by

$$T_m : X \cong \mathbb{P}^1 \to \mathbb{P}^1.$$

This cover has type $(m, n, p) = (m, 1, 2)$; it is branched over the three points $(b_1, b_2, b_3) = (-1, 1, \infty)$ with monodromy $[rf]$, $[f]$ and $[r]$ respectively. Since there is an essentially unique action of D_{2m} on $\mathbb{P}^1$, the Chebyshev polynomial gives the unique dihedral cover of type $(m, 1, 2)$. (However there are no dihedral forms when $p = 2$, since $g(X) = 0$.)

The case $p = 4$. The Veech and Ward example arise from dihedral covers with $p = 4$. These covers are obtained by pulling back the canonical dihedral cover with $p = 2$ under a suitable polynomial map $P : \mathbb{P}^1 \to \mathbb{P}^1$.

The Veech examples. Let us begin with the Veech examples, which are associated to triangles of type $(1, 1, m-2)$. Given $m \geq 3$, let

$$V = \{t \in \mathbb{C} \,:\, t(t^{2m} - 1) \neq 0\}.$$

Consider the family of Riemann surfaces X_t and Y_t, defined for $t \in V$ by

$$\begin{aligned} x^2 - 1 &= t^{-m} T_m(ty) \quad \text{and} && \text{(A.4)} \\ x^2 - 1 &= t^{-m} T_m(C(z)) = t^{-m} C(z^m) && \text{(A.5)} \end{aligned}$$

respectively. Then D_{2m} acts on Y_t via (A.3), and the quotient map $\xi : Y_t \to X_t$ is given by $y = t^{-1}C(z)$. The induced degree m map $\pi_t : X_t \to \mathbb{P}^1 = Y_t / D_{2m}$ is given by $\pi(x, y) = x$. It is easily verified that X_t has genus $g = \lfloor (m-1)/2 \rfloor$. Let

$$\omega_t = \frac{dy}{x} \in \Omega(X_t).$$

Proposition A.2 *We have $(X_t, \omega_t) \in \Omega D_{m,2,4}$ for all $t \in V$.*

Proof. The equation for X_t has the form $P(x) = Q_t(y)$, where $P(x) = x^2 - 1$ and

$$Q_t(y) = t^{-m} T_m(ty)$$

is a monic polynomial of degree m in y. The critical values of P and Q_t are $\{-1\}$ and $\{\pm t^{-m}\}$ respectively, so P and Q_t have no common critical values for $t \in V$. Thus $(\omega_t) = v \cdot X_t^*$, since it is an instance of equation (A.1).

Note that the equation $x = Q_t(y)$ presents $\mathbb{P}^1_y$ as a dihedral cover of $\mathbb{P}^1_x$ of type $(m, 1, 2)$. Thus for $t \in V$, $P(x) = Q_t(y)$ defines a dihedral cover $\pi_t : X_t \to \mathbb{P}^1_x$ of type $(m, 2, 4)$. Indeed, its five branch points are given by

$\{b_1, b_2\} = P^{-1}(t^{-m})$, with monodromy $[f]$; by $\{b_3, b_4\} = P^{-1}(-t^{-m})$, with monodromy $[rf]$; and by $b_5 = \infty$, with monodromy $[r^2]$ (since $\deg(P) = 2$).

To check that ω is an eigenform for $T = r + r^{-1}$ acting on $\mathrm{Jac}(Y)$, we observe that the change of variables $y = t^{-1}C(z)$ gives

$$\omega = t^{-1}S(z)\frac{dz}{z},$$

where $S(z) = (z - z^{-1})/2$. Hence the identity $S(\alpha z) + S(\alpha^{-1})(z) = (\alpha + \alpha^{-1})S(z)$ implies that $T^*\omega = (\zeta_m + \zeta_m^{-1})$. ■

Proof of Theorem A.1, Veech case. Observe that equation (A.4) defines X_t as a hyperelliptic curve. Since the cross–ratios of the zeros of Q_t (together with ∞ when m is odd) vary with t, the image ΩV of the map

$$\mathbb{C}^* \times V \to \Omega D_{m,2,4}$$

given by $(s, t) \mapsto (X_t, s\omega_t)$ is two dimensional. By Theorem 3.3, ΩV is linear in period coordinates, and hence its closure is $\mathrm{SL}_2(\mathbb{R})$–invariant.

Now observe that as $t \to \infty$, the polynomial $Q_t(y)$ converges to y^m. Hence (X_t, ω_t) converges to the form $\omega_\infty = dx/y$ on the curve X_∞ defined by $y^m = P(x) = x^2 - 1 = (x-1)(x+1)$. Since the zeros of P are simple, $(X_\infty, \omega_\infty)$ is a cyclic form of type $a = (1, 1, m-2)$, and such a form is unique up to a scalar multiple. ■

The Ward examples. We now turn to the Ward examples, which are associated to triangles of type $(1, 2, m-3)$ with m even.

Given an even integer $m \geq 4$, consider the family of Riemann surfaces X_t defined by

$$x^3 - x^2 = t^{-m}(T_m(ty) + 1), \tag{A.6}$$

and let $\pi_t : X_t \to \mathbb{P}^1$ be given by $\pi_t(x, y) = x$. Equation (A.6) has the form $P(x) = Q_t(y)$, where the critical values of P and Q_t are $\{0, -4/27\}$ and $\{0, 2t^{-m}\}$ respectively. Let $V \subset \mathbb{C}^*$ denote the set of t such that $2t^{-m} \neq -4/27$, so that $z = 0$ is the only common critical value of P and Q_t. Then for all $t \in V$, X_t has genus $g = m/2$, and the form

$$\omega_t = \frac{x\,dy}{P'(x)}$$

satisfies $(\omega_t) = v \cdot X_t^*$, as it is an instance of equation (A.2).

Proposition A.3 *We have $(X_t, \omega_t) \in \Omega D_{m,3,4}$ for all $t \in V$.*

Proof. The map $\pi_t : X \to \mathbb{P}^1$ is the pullback of the dihedral map $x = Q_t(y)$, which has critical values $(0, 2t^{-m}, \infty)$. Thus π_t has branch points with monodromy $[f]$ at the three points $\{b_1, b_2, b_3\} = P^{-1}(2t^{-m})$. There is also potential branching at the points $\{b_4, b_4'\} = \{1, 0\} = P^{-1}(0)$. Indeed, there is branching with monodromy $[rf]$ at the simple zero $b_4 = 1$ of $P(x)$, but there is no branching at the double zero $b_4' = 0$, since $[rf]$ has order 2. Finally the monodromy about $b_5 = \infty$ is $[r^3]$, since $\deg(P) = 3$. Thus π_t is a dihedral map of type $(m, n, p) = (m, 3, 4)$, and the form ω_t is dihedral because its divisor is a multiple of X_t^* and $(r + 1/r)^*\omega = \tau_m\omega$ (by the same reasoning as in the Veech case). ∎

Proof of Theorem A.1, Ward case. When the nonzero critical values of $P(x)$ and $Q_t(y)$ collide, the curve X_t acquires a node and thus $X_t \to \infty$ in $\mathcal{M}_g$. It follows that the map $V \to \mathcal{M}_g$ given by $t \mapsto X_t$ is non-constant, so its image is one dimensional. Consequently, the subvariety V of $\Omega D_{m,3,4}$ arising from forms of the type $(X_t, s\omega_t)$, $(s, t) \in \mathbb{C}^* \times V$ is irreducible and 2–dimensional. By Theorem 3.3, V is linear in period coordinates, and hence its closure is $\mathrm{SL}_2(\mathbb{R})$–invariant.

As in the Veech examples, the polynomial $Q_t(y)$ converges to y^m as $t \to \infty$. Hence (X_t, ω_t) converges to the form $\omega_\infty = x\,dy/P'(x)$ on the curve X_∞ defined by $y^m = P(x) = x^2(x-1)$. This is a cyclic of type $a = (1, 2, m-3)$; indeed, if we make the change of variables $u = x(x-1)/y$, then we have

$$u^m = x^{m-2}(x-1)^{m-1},$$

and

$$\omega_\infty = \frac{x\,dy}{P'(x)} = \frac{x\,dx}{my^{m-1}} = \frac{xy\,dx}{mx^2(x-1)} = \frac{dx}{m \cdot u}\cdot$$

There is a unique cyclic form of this type, up to scale, and the theorem follows. ∎

Remark: the case $p \geq 6$. Most dihedral covers with $p \geq 6$ are not directly related to the dihedral action of D_{2m} on $\mathbb{P}^1$. For example, the dihedral cover $\widetilde{\pi} : Y \to \mathbb{P}^1$ with $p = 6$ defined by equation (4.7) is a pullback of the cover $x = C(y^m)$ if and only if the polynomial $q(x)$ is a square.

B Pentagons

In this section turn from quadrilaterals to pentagons and prove Theorem 1.6. Namely, we show that the cyclic forms of type $a = (1, 1, 2, 2, 12)$ generate a primitive, $\mathrm{SL}_2(\mathbb{R})$–invariant variety $\Omega G_a \subset \Omega\mathcal{M}_4$ of dimension 6. The corresponding dihedral forms have type $(m,n,p) = (6,6,8)$ by equation (3.12).

A typical pentagon of type a is shown in Figure 4. Note that the 'boundary' makes a complete turn at its internal vertex, where the angle is $12\pi/6 = 2\pi$. When the length of the internal edge shrinks to zero, this pentagon degenerates to a quadrilateral of type $(1,1,2,8)$; this type already appears in Table 1.

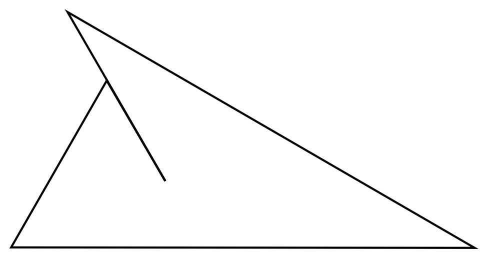

Figure 4. A pentagon of type $(1,1,2,2,12)$.

The main step in the proof of Theorem 1.6 is to extend the results of §4 to show:

Theorem B.1 *For $(m,n,p) = (6,6,8)$, the locus $\Omega D_{m,n,p} \subset \Omega\mathcal{M}_4(1^6)$ is an irreducible, unirational variety of dimension 6. Its closure contains the variety ΩZ_a where $a = (1,1,2,2,12)$.*

Assuming this for now, we can deduce:

Corollary B.2 *The closure of the dihedral locus is generated by pentagons: we have*

$$\overline{\Omega D}_{m,n,p} = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot \Omega Z_a} = \Omega G_a$$

when $a = (1,1,2,2,12)$ and $(m,n,p) = (6,6,8)$.

Proof. The proof follows the same lines as the proof of Theorem 5.4 in §5, using the fact that

$$\dim D(X,\omega,T) = \dim \Omega D_{m,n,p} = 2 \dim \Omega Z_a$$

by Theorem 3.3, the result above, and Theorem 2.2. The first equality implies that $\overline{\Omega D}_{m,n,p}$ is $\mathrm{SL}_2(\mathbb{R})$–invariant, so it contains ΩG_a, and the second implies that it equals ΩG_a: since the latter is locally defined by *real* linear equations in period coordinates [EMM], its dimension is at least twice that of ΩZ_a. ■

Proof of Theorem 1.6. The locus $\Omega G_a \subset \Omega \mathcal{M}_4$ is $\mathrm{SL}_2(\mathbb{R})$–invariant by construction, it is 6–dimensional by the two results above, and it is primitive because its closure contains the cyclic forms of type $(1,1,2,8)$; a generic form of this type was shown to be primitive in the proof of Theorem 5.4. ■

Sketch of the proof of Theorem B.1. The proof of this result follows closely the quadrilateral case $N = 4$, which was treated in §4. We simply extend the argument to cover the case $N = 5$ and $(m,n,p) = (6,6,8)$. (Recall $p = 2N - 2$.)

In fact, most of the argument goes through for general $N \geq 3$. The main modification needed in the definitions concerns the space of pairs of polynomials $P_{m,n,p}$: the condition $\deg(d_1 d_2) = 6$ in equation (4.5) need to be replaced by

$$\deg(d_1 d_2) = 2g(C) + 2 = p,$$

and similarly for $\deg(D)$ in equation (4.3). The proofs of Theorem 4.2, Proposition 4.15 and Theorem 4.4 can then be adapted to show we have $\Omega Z_a \subset \overline{\Omega D}_{m,n}$, that $\Omega D_{m,n}$ is unirational, and that

$$\dim V_{m,n,p} = \frac{p-2}{2} + \lfloor 2n/m \rfloor. \tag{B.1}$$

Here $V_{m,n,p} \subset \mathcal{M}_{0,p+1}$ is the space of configurations $B \subset \mathbb{P}^1$ that arise as the critical values of a dihedral covers $\pi : X \to \mathbb{P}^1$ of type (m,n,p).

It remains to show that, for $m \leq 2g(X) - 2$, we have:

$$\dim \Omega D_{m,n,p} = \frac{p}{2} + \lfloor 2n/m \rfloor. \tag{B.2}$$

We will show this equation holds under the assumption that

$$N = \epsilon(m,n) + 4. \tag{B.3}$$

(Recall $\epsilon(m,n) = 1$ if $m|n$ and 0 otherwise.) First note that condition (B.3) implies, by Theorem 3.3, that any dihedral triple (X, ω, T) of type (m,n,p) satisfies

$$\dim \operatorname{Ker}((T^* - \tau_m)|\Omega(X)) = 2.$$

It follows that the associated dihedral map $\pi : X \to \mathbb{P}^1$ is given by $\pi(x) = \omega'(x)/\omega(x)$, where (ω, ω') form a basis for $\operatorname{Ker}(T^* - \tau_m)$ (see the proof of

Proposition 4.19). This implies that a given form $(X, \omega) \in \Omega D_{m,n,p}$ is associated to at most finitely many $B \in V_{m,n,p}$. Taking into account the fact that scaling ω does not change B, we obtain equation (B.2).

Since the condition (B.3) holds for $(m, n, p) = (6, 6)$, equation (B.2) gives $\dim \Omega D_{6,6,8} = 6$. ■

Remarks.

1. The pentagon variety ΩG_a is defined over $\mathbb{Q}$ since $m = 6$, and it has rank two by Theorem 3.3. Thus the construction in §7 applies, to yield a dense set of 4–dimensional, $\mathrm{SL}_2(\mathbb{R})$–invariant subvarieties $\Omega G_{a,D} \subset \Omega G_a$.

2. The intersection of the pentagon variety with the stratum $\Omega\mathcal{M}_4(1^2, 2^2)$ yields a new invariant variety of rank two and dimension five. To see this variety is nonempty, observe that the dihedral map $\pi : X \to \mathbb{P}^1$ associated to a point in $\Omega D_{6,6,8}$ is unbranched over $b_{p+1} = b_9$. By holding $(b_1, \ldots, b_8)$ fixed and letting b_9 collide with one of the other points, one can make the zeros of ω collide as well.

3. The proof of Theorem B.1 also shows that the locus $\Omega D_{4,8,8} \subset \Omega\mathcal{M}_3(1^6)$ has dimension six. In this case, however, the space of dihedral forms is already known: it is obtained from $Q\mathcal{M}_{1,4}(-1^4, 2^2)$ via a covering construction.

4. The six–dimensional locus $\Omega D_{6,6,8}$, on the other hand, is a proper subvariety of seven–dimensional locus arising from double covers of differentials in $Q\mathcal{M}_{2,2}(-1^2, 2^3)$, so it does not come from a stratum of one forms or quadratic differentials.

5. We expect that equation (B.2) holds for most values of (m, n, p) with $m \leq |\chi(X)|$.

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