THE ASYMPTOTIC NUMBER OF UNLABELLED REGULAR GRAPHS

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Over ten years ago Wright [4] proved a fundamental theorem in the theory of random graphs. He showed that if $M = M(n)$ is such that almost no labelled graph of order $n$ and size $M$ has two isolated vertices or two vertices of degree $n - 1$, then the number of labelled graphs of order $n$ and size $M$ divided by the number of unlabelled graphs of order $n$ and size $M$ is asymptotic to $n!$. The result is best possible, since if the above ratio is asymptotic to $n!$ then almost no labelled graph of order $n$ and size $M$ has a non-trivial automorphism. The aim of this paper is to prove the analogue of Wright's theorem for regular graphs.

Random regular graphs have not been studied for long. The main reason for this is that until recently there was no asymptotic formula for the number of labelled regular graphs; such a formula was found by Bender and Canfield [1]. Even more recently, in [2] a model was given for the set of labelled regular graphs which makes the study of random regular graphs fairly accessible. In particular, a result in [3] implies that, for $r \geq 3$, almost every labelled $r$-regular graph has only the identity as its automorphism.

Let $r \geq 3$ be fixed and let $n \to \infty$ in such a way that $rn = 2m$ is even. Denote by $\mathcal{L}_r = \mathcal{L}_r(n)$, the set of $r$-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. Write $\mathcal{U}_r = \mathcal{U}_r(n)$ for the set of unlabelled $r$-regular graphs of order $n$. Put $L_r = |\mathcal{L}_r|$ and $U_r = |\mathcal{U}_r|$. We know from [1] that

$$L_r \sim e^{-\frac{(r^2-1)4}{4}} \frac{(2m)!}{2^m m!} (r!)^{-n}.$$ 

The aim of this paper is to prove the following result.

**Theorem.** If $r \geq 3$ then

$$U_r \sim L_r / n! \sim e^{-\frac{(r^2-1)4}{4}} \frac{(2m)!}{2^m m!} (r!)^{-n}/n!,$$  \hfill (1)

where $m = rn/2$.

**Proof.** The second relation is the result of Bender and Canfield [1] and so we have to prove that $U_r \sim L_r / n!$. Throughout the proof all relations ($\sim$, $o$, $o$) refer to the passage of $n$ to $\infty$ and our inequalities are claimed to hold if $n$ is sufficiently large. We shall write $c_1, c_2, \ldots$ for positive constants.

Let $W = \bigcup_{i=1}^{n} W_i$ be a fixed set with $rn$ elements partitioned into $n$ sets of $r$ elements each. Call a partition of $W$ into $m = rn/2$ pairs a configuration. Clearly there are $N(m) = (2m)! / 2^m m!$ configurations. Given a configuration $F$, denote by $\phi(F)$ the graph with vertex set $V = \{1, 2, \ldots, n\}$ in which $ij$ is an edge if and only if $F$

Received 14 September, 1981.

has a pair with one element in \( W_i \) and the other in \( W_j \). Clearly every \( G \in \mathscr{L}_r \) is of the form \( G = \phi(F) \) for \( (r!)^n \) configurations \( F \), so that \( L_r \leq N(m)/(r!)^n \). What we shall need from the asymptotic formula for \( L_r \) is that if \( n \) is sufficiently large then

\[
L_r \geq c_1 N(m)/(r!)^n
\]

for some positive constant \( c_1 \). (In fact every \( 0 < c_1 < e^{-(r^2-1)/4} \) will do.)

It is convenient to turn \( \mathscr{L}_r \) into a probability space by giving all graphs \( G \in \mathscr{L}_r \) the same probability, namely \( L_r^{-1} \). Let us estimate the probability that a random graph \( G \in \mathscr{L}_r \) contains a fixed set of \( u \) edges. Clearly there are at most \( r^{2u} \) sets of \( u \) pairs that are mapped by \( \phi \) into our \( u \) edges. There are \( N(m-u) \) configurations that contain \( u \) given pairs, and so the above probability is at most

\[
P_u = \frac{r^{2u}N(m-u)}{c_1 N(m)} = \frac{1}{c_1} \frac{r^{2u}N(m-u)}{N(m)}
\]

\[
= \frac{1}{c_1} \frac{r^{2u}(2m-1)(2m-3)\ldots(2m-2u+1)}{(r^2 - 1)!}^{-1}
\]

\[
\leq \frac{1}{c_1} r^{2u}((2m-2)!)^{-u/(2m-1)} \leq \left( \frac{c_2}{n} \right)^u.
\] (2)

Let us now see what we have to show to prove (1). Consider the symmetric group \( \Sigma_n \) acting on \( V \). For \( \omega \in \Sigma_n \) let \( \mathscr{L}(\omega) \) be the set of graphs in \( \mathscr{L}_r \) invariant under \( \omega \) and put \( L(\omega) = |\mathscr{L}(\omega)| \). Then

\[
\sum_{\omega \in \Sigma_n} L(\omega) = \sum_{G \in \mathscr{L}_r} a(G),
\]

where \( a(G) \) is the order of the automorphism group of \( G \). Since \( \mathscr{L}_r \) has precisely \( n!/a(G) \) graphs isomorphic to a given graph \( G \in \mathscr{L}_r \),

\[
U_r = \sum_{G \in \mathscr{L}_r} (n!/a(G))^{-1} = \frac{1}{n!} \sum_{G \in \mathscr{L}_r} a(G) = \frac{1}{n!} \sum_{\omega \in \Sigma_n} L(\omega).
\]

For the identity permutation \( 1 \in \Sigma_n \) we have \( L(1) = L_r \) so that (1) holds if and only if

\[
\sum_{\omega \in \Sigma_n} L(\omega)/L_r = \sum_{\omega \in \Sigma_n} P(\omega) = o(1),
\] (3)

where \( P(\omega) = L(\omega)/L_r \) is the probability that \( G \in \mathscr{L}_r \) is invariant under \( \omega \).

We partition the sum in (3) into several parts. Denote by \( M(s,s_2,s_3) \) the set of permutations in \( \Sigma_n \) that move \( s \) vertices and have \( s_2 \) 2-cycles and \( s_3 \) 3-cycles. Note that if \( M(s,s_2,s_3) \neq \emptyset \) then \( 2s_2 + 3s_3 \leq s \). Furthermore, \( M(0,0,0) = \{1\} \), \( M(1,0,0) = \emptyset \) and, rather crudely,

\[
|M(s,s_2,s_3)| \leq n!/\{s_2!s_3!\}.
\] (4)

The gist of our proof of (1) is an appropriate upper bound on \( \max \{ P(\omega) : \omega \in M(s,s_2,s_3) \} \). Though the proof of this upper bound is rather simple,
unfortunately it is somewhat cumbersome to present. We prepare the ground with a
simple but useful remark concerning representations of sets as unions of other sets.

Let \( \mathcal{A} \) be a collection of disjoint finite sets containing \( m_j \) sets of size \( j \),
\( j = 1, 2, \ldots, t \). Denote by \( f(b; \mathcal{A}) = f(b; m_1, m_2, \ldots, m_t) \) the number of \( b \)-sets that
are unions of some of the sets in \( \mathcal{A} \). Then

\[
f(b; 0, 0, \ldots, 0, m_h, m_{h+1}, \ldots, m_t) \leq f(b; b-h[b/h], 0, 0, \ldots, 0, m_h) = \left( \frac{m_h}{b/h} \right)^t \tag{5}
\]

where \( m'_h = \left[ \sum_i^t m_i / h \right] \).

The equality is obvious, and so we prove the inequality. Let \( \mathcal{A} \) be a collection of
disjoint sets consisting of \( m_j \) \( j \)-sets for \( h \leq j \leq t \), but containing no \( j \)-sets for \( j < h \).
For each \( A_i \in \mathcal{A} \) let \( C_i \) be an \( h \)-set so that \( \mathcal{C}' = \{ C_i : A_i \in \mathcal{A} \} \) consists of disjoint
\( h \)-sets. Let \( \mathcal{C}'' \) be a collection of \( m'_h - \sum_i^t m_j \) disjoint \( h \)-sets and \( b-h[b/h] \) singletons
so that \( \mathcal{C} = \mathcal{C}' \cup \mathcal{C}'' \) consists of disjoint sets. We have to show that
\( f(b; \mathcal{A}) \leq f(b; \mathcal{C}) \). To see this, suppose that a \( b \)-set \( B \) is the union of some sets in \( \mathcal{A} \).
Let \( B' \) be a \( b \)-set which is the union of those \( C_i \) for which \( A_i \subset B \) and some sets in \( \mathcal{C}'' \).
Then the map \( B \to B' \) is one-to-one and so \( f(b; \mathcal{A}) \leq f(b; \mathcal{C}) \), proving (5).

Let us consider the orbits of (the permutation induced by) \( \omega \) acting on those
elements of \( V^{(2)} \), the set of unordered pairs of elements of \( V \), that have at least one
element in \( S \). For simplicity we say that an unordered pair \( (x, y) \in V^{(2)} \) joins \( S_k \) to \( S_l \) if
\( x \in S_k \) and \( y \in S_l \).

Denote by \( m_h(k, l) \) the number of orbits of size \( h \) consisting of pairs joining \( S_k \) to
\( S_l \). By definition \( m_h(k, l) = m_h(l, k) \). It is easily checked that

\[
\begin{align*}
m_h(k, 1) &= 0 \quad \text{unless } h \geq h(k, 1) = k, & m_h(3, 4) &= 0 \quad \text{unless } h \geq h(3, 4) = 6, \\
m_h(2, 3) &= 0 \quad \text{unless } h = h(2, 3) = 6, & m_h(2, 2) &= 0 \quad \text{unless } h = h(2, 0) = 1, \\
m_h(2, 4) &= 0 \quad \text{unless } h \geq h(2, 4) = 4, & \quad & \quad \text{or } h = h(2, 2) = 2, \\
m_h(3, 3) &= 0 \quad \text{unless } h = h(3, 3) = 3, & m_h(4, 4) &= 0 \quad \text{unless } h \geq h(4, 0) = 2.
\end{align*}
\]

Furthermore,

\[
m_1(2, 2) \leq s_2 \quad \text{and} \quad 2m_2(4, 4) + 3m_3(4, 4) \leq 2s_4.
\]

To see the last statement note that if \( x \) is in a \( k \)-cycle but \( xy \) belongs to an orbit of
size less than \( k \), then \( k = 2l \) and \( y = \omega^l x \), that is, the pair \( (x, y) \) joins diametrically
opposite vertices of an even cycle.

In order to give the notation some consistency we write \( h(3, 0) = 1 \) and
\( h(4, 4) = 4 \). For \( 1 \leq k \leq l, l \neq 2 \), a \((k, l)\)-orbit (or \((l, k)\)-orbit) is an orbit of size at
least \( h(k, l) \) consisting of pairs joining \( S_k \) to \( S_l \). Furthermore, for \( k \geq 2 \) a \((k, 0)\)-orbit
is an orbit of size less than \( h(k, k) \) consisting of pairs joining \( S_k \) to \( S_k \). The 
\((k, 0)\)-orbits are the exceptional orbits of pairs joining vertices of \( S_k \) in the sense that 
altogether they contain at most \( k s_k/2 = |S_k|/2 \) edges, while the \((k, k)\)-orbits have at 
least \( k s_k (k s_k - 2)/2 \) edges.

Now we turn to our task of estimating \( P(\omega) \), \( \omega \in M(s_1, s_2, s_3) \). If \( G \in L_\omega(\omega) \) then the set of edges of \( G \) cannot split orbits. Suppose that \( G \) has \( h(i, j)k_{ij} \) edges in the set 
of \((i, j)\)-orbits. We call \((k_{ij})\) the set of parameters of \( G \). Note that these parameters 
have the following properties:

\[
k_i = k_j,
\]

\[
h(i, j)k_{ij} \text{ is a non-negative integer,}
\]

\[
\sum_{j=0}^{4} h(i, j)k_{ij} + h(i, 0)k_{i0} + h(i, i)k_{ii} = ris_i, \quad i = 2, 3, 4.
\]

The last relation holds since an edge in an \((i, j)\)-orbit contributes 1 to the sum of 
degrees of the vertices in \( S_i \) unless \( j = 0 \) or \( i \), when it contributes 2.

We shall call an array \((k_{ij})\) having the above properties a permissible array. When 
(6) is written out in full, a permissible array \((k_{ij})\) satisfies

\[
2k_{20} + 2k_{21} + 4k_{22} + 6k_{23} + 4k_{24} = 2rs_2,
\]

\[
3k_{31} + 6k_{32} + 6k_{33} + 6k_{34} = 3rs_3,
\]

\[
4k_{40} + 4k_{41} + 4k_{42} + 6k_{43} + 8k_{44} = 4rs_4.
\]

Given a permissible array \((k_{ij})\) we write \( P(\omega ; (k_{ij})) \) for the probability that a graph 
\( G \in L_\omega \) belongs to \( L(\omega) \) and has \((k_{ij})\) as its set of parameters. Then

\[
P(\omega) = \sum P(\omega ; (k_{ij}))
\]

where the summation is over all permissible arrays.

Let us estimate \( P(\omega ; (k_{ij})) \) for a fixed permissible array \((k_{ij})\). Denote by \( s(i, j) \) the 
number of pairs in the \((i, j)\)-orbits of \( \omega \). Clearly

\[
s(i, j) \leq 12s_is_j
\]

with \( s_0 \) defined to be 1. The set of \( h(i, j)k_{ij} \) edges consisting of full \((i, j)\)-orbits can be 
selected in at most

\[
f(i, j) = f(h(i, j)k_{ij} ; m_1(i, j), m_2(i, j), ..., m_t(i, j)) \leq \left( \frac{s(i, j)/h(i, j)}{[k_{ij}]} \right)
\]

ways. Here the inequality is a consequence of (5) and the fact that \( m_l(i, j) = 0 \) for 
\( l < h(i, j) \). By (2) the probability of a graph \( G \in L_\omega \) containing a given set of

\[
k = \sum h(i, j)k_{ij}
\]

\[
= k_{20} + 2k_{21} + 2k_{22} + 6k_{23} + 4k_{24} + 3k_{31} + 3k_{33} + 6k_{34} + 2k_{40} + 4k_{41} + 4k_{44}
\]
edges incident with $S$ is at most $(c_2/n)^k$. Hence

$$P(\omega ; (k_{ij})) \leq (c_2/n)^k \prod f(i,j),$$

where the product is over all unordered pairs $(i,j)$ with $\max(i,j) \geq 2$.

We should like to bound from above the right-hand side of (9) as $(k_{ij})$ runs over all permissible arrays. Though this is an elementary problem in optimization theory, the manipulations in a standard solution are offputting so we prefer a different approach.

We claim that

$$(c_2/n)^k \prod f(i,j) \leq n^{-3(1+\epsilon)} s_2^2 s_3^3$$

where $\epsilon = 2^{-4}$. To prove (10) we shall separate the large elements $k_{ij}$ from the small ones. Set

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } k_{ij} \geq 10^{-4} s/r, \\ 0 & \text{if } k_{ij} < 10^{-4} s/r. \end{cases}$$

Then

$$(c_2/n)^k \{\prod f(i,j)\} n^{1-10^{-2} s_2^{-2} s_3^{-3}} \leq n^{1-k} n \sum \varepsilon_{ij} k_{ij} \sum \varepsilon_{ij} k_{ij} s_2^{-2} s_3^{-3} = A,$$

say, where $\sum \varepsilon_{ij} k_{ij} = \sum_{j=2}^4 \varepsilon_{ij} k_{1j}$ and the summation in the exponent of $s$ is over $2 \leq i \leq j \leq 4$.

Recall that $s = 2s_2 + 3s_3 + 4s_4$ so that by (6') and (8) we have

$$s-k = k_{20} \left( \frac{2}{r} - 1 \right) + k_{21} \left( \frac{2}{r} - 2 \right) + k_{22} \left( \frac{4}{r} - 2 \right)$$

$$+ k_{23} \left( \frac{12}{r} - 6 \right) + k_{24} \left( \frac{8}{r} - 4 \right) + k_{31} \left( \frac{3}{r} - 3 \right) + k_{33} \left( \frac{6}{r} - 3 \right)$$

$$+ k_{34} \left( \frac{12}{r} - 6 \right) + k_{40} \left( \frac{4}{r} - 2 \right) + k_{41} \left( \frac{4}{r} - 4 \right) + k_{44} \left( \frac{8}{r} - 4 \right).$$

Hence if $s = n^r$, by (6') we find that

$$\log A \leq c_3 \frac{s}{\log n} + s-k + \sum_{j=2}^4 \varepsilon_{ij} k_{1j} + \alpha \sum_{2 \leq i < j \leq 4} \varepsilon_{ij} k_{ij}$$

$$- \alpha \sum_{j=0}^4 \varepsilon_{2j} h(2,j) k_{2j}/(2r) - \alpha \sum_{j=0}^4 \varepsilon_{3j} h(3,j) k_{3j}/(3r)$$

$$\leq c_3 \frac{s}{\log n} - k_{20} \frac{r-2}{r} - k_{21} \left( \frac{2}{r} - 1 \right) - k_{22} \left( \frac{2}{r} - \alpha + \frac{2}{r} \alpha \right)$$

$$- k_{23} \left( \frac{6}{r} - \varepsilon_{23} \alpha \left( 1 - \frac{3}{r} - \frac{2}{r} \right) \right) - k_{24} \left( \frac{4}{r} - \alpha + \frac{2}{r} \alpha \right)$$
\[ -k_{31} \left( \frac{3}{r} - \frac{1}{r} - 1 \right) - k_{33} \left( \frac{3}{r} - \frac{2}{r} - \alpha + \frac{\alpha}{r} \right) - k_{34} \left( \frac{6}{r} - \alpha + \frac{2\alpha}{r} \right) \]

\[ -k_{40} \frac{2r - 4}{r} - k_{41} \left( \frac{4}{r} - \frac{1}{r} - 1 \right) - k_{44} \left( \frac{4}{r} - \frac{2}{r} - \alpha \right) \]

\[ \leq c_3 \frac{s}{\log n} - \frac{r - 2}{r} \sum k_{ij} \leq c_3 \frac{s}{\log n} - \frac{r - 2}{12} s \leq s/13. \]

This implies (10) since \(1/13 - 10^{-2} > 2^{-4}\).

Now the proof is easily completed. For a permutation \(\omega \in M(s, s_2, s_3)\) there are at most \((6rs)^12\) permissible arrays and so by (7), (9) and (10)

\[ P(\omega) \leq (6rs)^12 n^{-s(1 + \epsilon)} s^2 s_2 s_3. \]

Therefore by (4)

\[ \sum_{\omega \in M(s, s_2, s_3)} P(\omega) \leq c_4 n^{-\epsilon s}. \]

\[ \sum_{\omega \in \Sigma_n, \omega \neq \epsilon s} P(\omega) \leq \sum_{s=2}^{n} s^2 c_4 n^{-\epsilon s} \leq \sum_{s=2}^{n} n^{-\epsilon s/2} = o(1). \]

This proves (3), and so the proof of our theorem is complete.

It is worth noting that the proof is considerably more pleasant for \(r \geq 5\). In that case we do not have to consider the number of 2-cycles and 3-cycles in a permutation \(\omega \in \Sigma_n\); it suffices to use the number of vertices moved by \(\omega\) as the only parameter, and to note that at most \(n^s\) permutations move precisely \(s\) vertices.

In this paper we considered regular graphs but in fact the proof can be adapted to graphs with more general degree sequences. In particular, the following result can be obtained without any additional work. Let \(\Delta \geq 3\) be a constant. Then the number of labelled graphs with degree sequences \(3 \leq d_1 \leq d_2 \leq \ldots \leq d_n \leq \Delta\) divided by the number of unlabelled graphs having the same degree sequence is asymptotic to \(n!\).

References


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