Homework 8

Math 55b
Due Tuesday, 31 Mar 2009.

1. Let f be a compactly support smooth function on $\mathbb{R}^2$. Are the relations $\int f(x,y)\,dx\,dy = \int f(y,x)\,dy\,dx$ and $dx\,dy = -dy\,dx$ both true? How can they be reconciled?

2. Prove that $\nabla \cdot v$ on $\mathbb{R}^3$ is the limit, as the size of a cube Q goes to zero, of the flux of v through ∂Q divided by the volume of Q. (Recall the flux is given by integrating $v \cdot n$ with respect to surface area, where n is the unit normal to ∂Q.)

3. State and prove a similar theorem for the three components of $\nabla \times v$ on $\mathbb{R}^3$.

4. Prove directly that $\nabla \cdot \nabla \times v = 0$ on $\mathbb{R}^3$. Then explain how this is a consequence of $d^2 = 0$.

5. How does the Hodge star on $\mathbb{R}^2$ operate on the differentials coming dr and $d\theta$ coming from polar coordinates? Use your answer to compute the Laplacian of a function $f(r,\theta)$ in polar coordinates. Then, find all radially symmetric functions $f(r)$ on $\mathbb{R}^2$ which are harmonic outside the origin.

6. Suppose α and β are forms of degree k and ℓ on $\mathbb{R}^n$. Prove a formula relating $\alpha\beta$ to $\beta\alpha$, and establish a 'product formula' for $d(\alpha\beta)$.

7. Give an example of an infinitely differentiable map $f : \mathbb{R} \to \mathbb{R}$ which is a homeomorphism but not a diffeomorphism.

8. For any smooth function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$, let
$$\frac{df}{dz} = \frac{1}{2}\left(\frac{df}{dx} - i\frac{df}{dy}\right) \quad \text{and} \quad \frac{df}{d\overline{z}} = \frac{1}{2}\left(\frac{df}{dx} + i\frac{df}{dy}\right).$$
(As usual $z = x + iy$.)

   (i) Prove that $df = (df/dz)\,dz + (df/d\overline{z})\,d\overline{z}$.

   (ii) Prove that $(d/dz)(z^n\overline{z}^m) = nz^{n-1}\overline{z}^m$.

   (iii) Prove that if
$$\sum_{0\le i,j\le N} a_{ij} z^i \overline{z}^j = \sum_{0\le i,j\le N} b_{ij} z^i \overline{z}^j$$
for all $z \in \mathbb{C}$, then $a_{ij} = b_{ij}$ for $0 \le i,j \le N$.

   (iv) Prove that a smooth function $f(z)$ is analytic iff $df/d\overline{z} = 0$, in which case $f'(z) = df/dz$.