Problems

Riemann Surfaces and Hyperbolic Geometry
Fall 2009

1. Let T be a triangle in the plane, and let $\mathcal{T}_n$ be the 6^n triangles obtained by repeated barycentric subdivision. Prove that most of the triangles in $\mathcal{T}_n$ are long and flat; more precisely, they have two angles near 0.

2. Let $G = \mathbb{Z} * \mathbb{Z}$ with generators a and b, and let $S = \{a, \overline{a}, b, \overline{b}\}$. Let μ be a probability measure on S and let $g_1, g_2, \ldots$ be the corresponding random walk on G. (i) Is it true that $g_n \to \infty$ for any measure μ? (ii) Construct a boundary for G and then show that, for suitable μ, g_n converges to a well-defined point on the boundary. (iii) Describe the stationary measure ν on the boundary. (iv) Show that ν_1 and ν_2 are mutually singular unless $\mu_1 = \mu_2$.

3. Continuing with $G = \mathbb{Z} * \mathbb{Z}$, define a useful notion of a 'harmonic function' on G, and show that every continuous function on ∂G extends to a harmonic function on G.

4. Define the notion of the 'visual extension' of a continuous function or vector field from S^{n-1}_∞ to $\mathbb{H}^n$. Show that (in the hyperbolic metric) the extended function is harmonic, and the extended vector field has zero divergence. Show that for $n = 2$, if the original vector field is quasiconformal, then its extension is quasi-isometric.

5. What is the right notion of a quasiconformal vector field on S^1?

6. Let $U \subset \widehat{\mathbb{C}}$ be the complement of 3 disks. Prove that U is conformally isomorphic to the complement V of 3 round disks.

7. Even better, show that a conformal map $f : U \to V$ can be constructed as the limit of maps $U_1 \to U_2 \to U_3 \to \cdots$, where U_{i+1} is obtained by applying the usual Riemann mapping theorem to U_i with 2 of its holes filled in.

8. Find a circle $C = S^1(z, r)$ in $\mathbb{C}$ such that C, together with the x and y axes, give the 3 sides of a hyperbolic $(2, 3, 7)$-triangle in the unit disk.

9. Prove that if $f : \Delta \to \Delta$ is a holomorphic map which is not a homeomorphism, then there exists a point $p \in \overline{\Delta}$ such that $f^n(z) \to p$ for all $z \in \Delta$.

   Give an example of a continuous map $f : \Delta \to \Delta$ which is strictly distance-decreasing in the hyperbolic metric, but does not have the property above.

10. Consider the ring $\mathrm{M}_2(\mathbb{R})$ as a vector space over $\mathbb{R}$. What are the signatures of the inner product $\langle A, B\rangle = \mathrm{tr}(A, B)$ and the quadratic form $q(A) = \det(A)$? How are these forms related?

    Show there is a natural involution $A \mapsto A^\dagger$ on V such that $\mathrm{tr}(A)I = A + A^\dagger$ and $\det(A)I = AA^\dagger$. Derive from this a useful formula for $\det(A + B)$.

    Show how to compute $\mathrm{tr}(A^2B)$ from $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, and $\mathrm{tr}(AB)$. What does this have to do with the representation variety of the free group into $\mathrm{SL}_2(\mathbb{R})$?

11. Given $X_i = \mathbb{H}/\Gamma_i$ in $\mathcal{T}_g$, let $\phi : S^1 \to S^1$ be the corresponding homeomorphism conjugating Γ_1 to Γ_2. Prove that if ϕ is C^1, then in fact ϕ is a Möbius transformation and hence $X_1 = X_2$.

12. Give an example of a representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ that *cannot* arise as the holonomy of a complex projective structure on a closed, orientable surface Σ.

13. Show there is a faithful representation $\rho : \mathbb{Z} * \mathbb{Z} \to \mathrm{Isom}\,\mathbb{H}^3$ such that the closure of the image is compact. Can this happen for $\mathbb{H}^2$?

14. Show there is no positive harmonic function on $\mathbb{R}^n$ for $n > 0$. Show there is a positive harmonic function on $\mathbb{R}^n - \{0\}$ for $n \geq 3$ but not for $n = 2$. What 'should' happen for $n = 2.5$?

15. Let $f(z) = \mathrm{Re}(1 + z)/(1 - z)$ be the positive harmonic function on the unit disk Δ corresponding the a δ-mass at $z = 1$ on the Martin boundary S^1. Let $p : \mathbb{H} \to \Delta^*$ be the covering map $p(z) = \exp(iz)$ to the punctured disk, and consider the positive harmonic function $h(z) = f(p(z))$ on $\mathbb{H}$. This function satisfies $h(z) = h(z + 1)$. Does it correspond to a period family of δ masses on the real axis, taking account of the fact that the Martin boundary of $\mathbb{H}$ is $\mathbb{R} \cup \{\infty\}$? (How

is this possible, since positive harmonic functions correspond to *finite* measures on the Martin boundary?)

16. Show that $f(x) = x_n^\alpha$ on $\mathbb{H}^n$ is an eigenfunction of the hyperbolic Laplacian on hyperbolic space, and compute its eigenvalue.

17. Prove that the (infinite) measure $dx\,dy/|x-y|^2$ on $\widehat{Q} \times \widehat{Q}$ is invariant under the diagonal action of $\mathrm{SL}_2(\mathbb{R})$. What form does this measure take on $S^1 \times S^1$?

18. Prove more generally that if μ is a conformal density of dimension δ on S^n, then $(\mu \times \mu)/|x-y|^{2\delta}$ is invariant under the Möbius group acting diagonally on $S^n \times S^n$. Here $|x-y|$ denotes the chordal distance, and μ is a measure satisfying $\mu(f(E)) = \int_E |f'|^\delta \mu$ for any conformal transformation f.

19. Prove the reverse Cauchy-Schwarz inequality: if $x, y \in \mathbb{R}^{n,1}$ and $x^2, y^2 < 0$, then $|\langle x, y\rangle|^2 \geq |x^2||y^2|$.

20. Prove the identity $2\cos(\alpha)\cos(\beta)\cos(\gamma)+\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma) = 1$ holds for any angles satisfying $\alpha + \beta + \gamma = \pi$.

21. Prove that for any convex subset $L \subset \mathbb{H}$, the minimum and maximum curvatures of $\partial B(L, R)$ tend to one (the curvature of a horocycle) as $R \to \infty$.

22. Let Y be a Galois covering of a compact Riemann surface X with deck group $\mathbb{Z}^n$. Prove that any bounded harmonic function on Y is constant. What about positive harmonic functions?

23. Let $P_r \subset \mathbb{H}^3$ be the polyhedron whose sides meet $\widehat{\mathbb{C}}$ in the lines $\mathrm{Re}\, z = \pm 1$, $\mathrm{Im}\, z = \pm 1$ and the circle $|z| = r$. For what values of r does P_r have finite volume? For what values of r do its dihedral angles lie in $\{\pi, \pi/2, \pi/3, \ldots\}$? When is the corresponding reflection group a lattice? When is it commensurable to $\mathrm{PSL}_2(\mathbb{Z}[i])$?

24. Draw a picture of the limit set of the group Γ_r generated by reflections in the sides of P_r, for $r = 1$ and for various other values of r. (Hint: google 'McMullen lim').

25. Let $d > 1$ be a square-free integer, let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-d}]) \subset \mathrm{Isom}^+(\mathbb{H}^3)$, and let $X = \mathbb{Q}(\sqrt{-d}) \cup \{\infty\}$. Show that X/Γ is a finite set, corresponding bijectively to the cusps (ends) of $\mathbb{H}^3/\Gamma$. In terms of d, how many cusps are there?

26. Let X be a hyperbolic surface of genus two built out of eight regular right pentagons with the 'obvious' gluing pattern. Give an equation describing X as a plane algebraic curve over $\mathbb{C}$. (One answer is $y^2 = p(x^2)$, where $p(x) = (x-1)(x^2 - \sqrt{5}x + 1)$.)

27. Find the Jacobian of X. (I.e. find an explicit lattice such that $\mathrm{Jac}(X) \cong \mathbb{C}^2/\Lambda$.)

28. Let $\Omega \subset \mathbb{R}^n$ be any domain other than $\mathbb{R}^n$ itself. Prove that Ω is complete in the metric $|dx|/d(x, \partial\Omega)$.

29. (Freedman) Prove that the Borromean rings cannot be realized by round circles.

30. Let $P \subset \mathbb{H}$ be an ideal quadrilateral. Choose isometries A and B that identify its sides in pairs. Show that for most choices of A and B, the resulting identification space $P/\sim$ is an *incomplete* hyperbolic manifold.

31. Prove that for $n = 1$, a quasiconformal homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is Hölder continuous (that is, $|\phi(x) - \phi(y)| \le C|x-y|^\alpha$ for some C and α). Generalize to $\mathbb{R}^n$.

32. Let $[p, q]$ denote the geodesic segment joining two random point $p, q \in S^1(x, R) \subset \mathbb{H}^2$, with $R \gg 0$. What is the expected value, roughly, of the distance $d(x, [p, q])$?

33. Let $Q \subset \widehat{\mathbb{C}}$ be a K-quasicircle, and let $d(Q)$ be the Hausdorff distance between the two boundary components of the convex hull of Q in hyperbolic space. (i) Prove that $d(Q) \le D(K)$. (ii) Prove the converse does not hold. More precisely, give an example of a nondegenerate continuum K which is a Hausdorff limit of a sequence of quasicircles with $\sup d(Q_n) < \infty$, but which is not itself a quasicircle.

34. Let $f, g : \mathbb{C} \to \mathbb{C}$ be smooth quasiconformal maps. Compute the complex dilatation $\mu(f \circ g)$ in terms of $\mu(f)$ and $\mu(g)$. Explain why the answer looks like a Möbius transformation.

35. Let $f : \Delta - E \to \mathbb{C}$ be a bounded holomorphic function, where E is a closed set of Hausdorff dimension < 1. Show that f extends to a holomorphic function on Δ, i.e. the singularities at E are removable.

36. Given an example of a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that f is holomorphic on an the ope set $\mathbb{C} - E$, the Hausdorff dimension of E is one, but f is not globally holomorphic.

37. Prove that the Murasugi sum of two fibers in S^3 gives another fiber. (Hint: think of the construction as giving a connected sum of two copies of S^3.)

38. Find the sequence of fibered knots and links coming from the A_n diagrams. By varying the direction of the twists, give both hyperbolic examples (like the figure 8) and examples with monodromy of finite order (like the trefoil).

39. Prove that the $(-2, 3, 7)$ knot is fibered, and relate it to the E_{10} diagram.

40. Let G be the Lie group of isometric motions of $\mathbb{R}^3$. Give an example of a sequence of cyclic groups $\Gamma_n = \langle \gamma_n \rangle \subset G$ which converge geometrically to a discrete subgroup $\Gamma \subset G$ isomorphic to $\mathbb{Z}^2$.

41. Let $(M_n, p_n) \to (M, p)$ be a geometrically convergent sequence of finite volume n-manifolds. Prove that $\mathrm{vol}(M) \leq \liminf \mathrm{vol}(M_n)$, and equality holds if $\dim M_n \geq 3$.

42. Let $K \subset M$ be the thick part of a hyperbolic n-manifold. Prove that $\pi_1(K)$ injects into $\pi_1(M)$ if $n \geq 4$.

43. Let M_i be a sequence of n-manifolds with closed geodesics of length $L_i \to 0$. Prove that if $n \geq 4$, then $\mathrm{vol}(M_i) \to \infty$. (*) Show by example that this is false for $n = 3$.

44. Let Σ be a surface of genus $g \geq 2$, and let $\mathcal{P}$ denote the countable set of pants decompositions of Σ (up to isotopy). Define the notation of an

elementary move $P \equiv P'$, and show that any two pants decompositions can be joined by a sequence of such moves.

45. Construct an explicit nontrivial cocycle in the group cohomology $H^2(\mathrm{Mod}_g, \mathbb{Z})$ (equivalently, construct a nontrivial central extension of Mod_g or a nontrivial circle bundle over $\mathcal{M}_g$).

46. Let P_n be a sequence of pairs of pants with basepoints in the thick part. What are the possible geometric limits of P_n?

47. Let $X = \mathcal{M}_g - \mathcal{M}_g(r)$, $r > 0$ denote the thin part of moduli space, for $g > 1$. Prove that $\pi_1(X)$ maps onto $\pi_1(\mathcal{M}_g)$.

48. Let X be a compact Riemann surface of genus $g \geq 1$, and let $B_X \subset Q(X)$ be the unit ball in the space of holomorphic quadratic differentials with respect to the norm $\|\phi| = \int_X |\phi|$.

    Prove that (i) B_X is strictly convex and (ii) B_X has a unique supporting hyperplane H_p at each point $p \in \partial B_X$.

    Prove, in fact, that ∂B_X is C^1 in the sense that the supporting hyperplane H_p varies continuously with p.