Math 122, Solution Set No. 5

1 5.2.13

(a) ($\Rightarrow$) if x is on l, then glide reflection acts on points on l as a translation; therefore, $x, m(x)$ and $m^2(x)$ lie on l and are colinear. ($\Leftarrow$) If x is not on l, then the line joining x to $m(x)$ crosses l (because m is a glide reflection) and so does the line from $m(x)$ to $m^2(x)$. If these points are distinct, then this implies $x, m(x), m^2(x)$ are not colinear, since any line intersects l at most once. (b) From Artin, if m is orientation-reversing, it is either a reflection or glide reflection.If x is a reflection, $m^2(x) = x$ and so m is a glide reflection; by (a) it is along the line l.

2 5.2.14

By Artin (2.4) etc., we know that any $m \in SM$ has the form $m = t_a \rho_\theta$ with $a \in \mathbb{R}^2 and \theta \in \mathbb{R}$. Write a as (a_1, a_2) and define the map $\varphi : SM \to GL_2(\mathbb{C})$ by

$$\varphi(t_a \rho_\theta) = \begin{bmatrix} e^{i\theta} & a_1 + ia_2 \\ 0 & 1 \end{bmatrix}$$

This is clearly a bijection between SM and the desired subgroup of $GL_c(\mathbb{C})$, and it is easily checked that it is a homomorphism and therefore an isomorphism. Note: we may not assume that if t_a is a translation, $a \in \mathbb{R}$.

3 5.3.3

Let $D_n = \langle x, y \rangle | x^n = y^2 = 1 and xy = yx^{-1}$

For D_{13}: by Lagrange, proper normal subgroups have order 2 or 13; the order 2 $\langle y \rangle$ subgroup is not normal, since $xyx^{-1} = 2x \notin \langle y \rangle$. The order 13 subgroup $\langle x \rangle$ has index 2, and is therefore normal; the quotient group has order 2 and is therefore C_2.

For D_{15}; we have the normal subgroups $\langle x^1 \rangle, \langle x^3 \rangle, \langle x^5 \rangle$ which are normal because they are the kernels of the homomorphisms $\varphi_n : D_{15} \to D_n, n = 1, 3, 5$ defined by $\varphi_n(x) = x, \varphi(y) = y$. The first Isomorphism Theorem tells us that the quotient groups are $D_n, n = 1, 3, 5$. All other subgroups contain the element y, which (as above) implies that they are not normal.

Note: the dihedral group is not only the symmetries of an n-gon, but also exists as an abstract group; computations can be made in the dihedral group and its structure analyzed without reference to rotations and reflections.

4 5.4.2

Let x be a rotation around point a and y a rotation around point b. Consider the motion $xyx^{-1}y^{-1}$. This is clearly nonzero, since rotations around different points do not commute. However, if we take the image of this motion under the derivative map $\varphi : M \to O_2(\mathbb{R})$, we see that it is sent to the identity, since the motions x and y are orientation-preserving and hence $\varphi(xyx^{-1}y^{-1}) \in SO_2(\mathbb{R})$, which is an abelian group. Therefore $xyx^{-1}y^{-1} \in \ker \varphi$, i.e. it is a translation.

5 5.4.4

Numbering the patterns 1-17, we have the following point groups: 1) C_1; 2) C_2; 3) C_3; 4) C_4; 5) C_6; 6) D_1; 7) D_1; 8) D_1; 9) D_2; 10) D_2; 11) D_2; 12) D_2; 13) D_4; 14) D_4; 15) D_3; 16) D_3; 17) D_6.

6 5.4.6

Note that $\sqrt{2}-1<1$, so for some n, $(\sqrt{2}-1)^n<\epsilon$ for any $\epsilon>0$. Furthermore, since $(\sqrt{2})^2=2$, by the binomial expansion this product is of the form $a+b\sqrt{2}$ with $a,b\in\mathbb{Z}$. So we can find elements of this subgroup arbitrarily close to zero, and therefore can add arbitrarily small elements to find an element of the subgroup arbitrarily close to any real number, i.e., the subgroup of $\mathbb{R}^+$ generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}$.

7 5.4.10

(a) this point group contains a $\pi/2$ rotation and a reflection, and is therefore isomorphic to D_2. (b) There are four cosets in the point group: they are (1) translations of the frieze pattern, (2) translations of the frieze pattern with a rotation, (3) translations with a reflection, and (4) translations with both a reflection and rotation. (c) $[G:G\cap T]=|G/G\cap T|=|\text{point group}|=4$.

8 5.4.12

As in 5.4.10, Note that $[G:G\cap T]=|\text{point group}|$. The point group of symmetries of an equilateral triangular lattice is not in fact D_3, but rather D_6 (as in class). Therefore $[G:G\cap T]=|D_6|=12$.

9 5.4.14

Many people had incorrect answers for parts of this problem. In addition to reading the solution set, it would be helpful to look at the Japanese wallpaper patterns on the course website. Numbering the patterns on page 190 as (a) through (f), and those on p. 173 as (1) through (17), we note the following; (a) has 90-degree rotational symmetry about the center of a square, but no reflectional symmetry; it is therefore has the same symmetry type as pattern (4). (b) has 90-degree rotational symmetry and reflections, but NO center of 90-degree rotation lies on an axis of reflection; therefore this is the same symmetry type as pattern (14). (c) This pattern has bilateral but not rotational symmetry, and its lattice is that of an equilateral triangle, as is pattern (8). (d) this pattern has a glide reflection and no other point symmetries, as does pattern (7). (e) has the same symmetries as pattern (b), and (14) on page 173. (f) has 180-degree rotational symmetry and translational symmetry but no others; likewise is pattern (2) on page 173.

10 5.5.5

Note that the group of symmetries of the square acts transitively on the vertices, edges, and diagonals of the square. The symmetry group has order 8, so using the counting formula we can immediately classify the stabilizer of an edge or vertex as C_2 - in the case of an edge, this corresponds to the identity and reflection about the perpendicular line through the center of that edge. In the case of a vertex, the identity and reflection about the diagonal line through that vertex. There are only two diagonals, so the stabilizer of a diagonal has 4 elements; the identity, rotation of 180 degrees, reflection about the diagonal, and the composition of the latter two elements.