1. For each description below,
   - sketch the graph of a cubic function $f(x)$ with the specified characteristics.
   - find a formula for $f(x)$

There may be many possible answers, or none at all.

(a) $f(x)$ has zeros at $x = -\pi$, $x = 2$, and $x = 5$.

(b) $f(x)$ has only two zeros, at $x = 2$ and $x = 5$, and a $y$-intercept of $-4$.

(c) $f(x)$ has only one zero, at $x = 2$, and passes through the point $(1,5)$.

(d) $f(x)$ has no zero, and passes through the points $(2,3)$, $(5,-1)$.

2. Sketch the graph of $y = x(x + 2)(x - 2)$. Where are the points having horizontal tangent lines?

3. Find a cubic polynomial with local minimum at $x = 2$ and local maximum at $x = 5$. Where is the inflection point?
4. For each description below,

- sketch the graph of a polynomial function $P(x)$ with the specified characteristics.
- find a formula for $P(x)$

*There may be many possible answers, or none at all.*

(a) A degree 4 polynomial with a zero of multiplicity two at $x = 1$, zeros at $x = 5$ and $x = e$, and a $y$-intercept of 7.

(b) A degree 5 polynomial with no zeros and $\lim_{x \to \infty} P(x) = -\infty$.

(c) A degree 6 polynomial with no zeros and $\lim_{x \to \infty} P(x) = -\infty$.

5. (a) How many zeros can a degree $n$ polynomial have at most? At least?

(b) How many points with horizontal tangent line can a degree $n$ polynomial have at most? At least?
Cubics and Higher-Degree Polynomials – Solutions

1. (a) \( f(x) = a(x + \pi)(x - 2)(x - 5) \)

(b) \( f(x) = a(x - 2)^2(x - 5) \) or \( f(x) = b(x - 2)(x - 5)^2 \). We have \( f(0) = -4 \), so \(-4 = a \cdot 4 \cdot (-5) \) and \(-4 = b \cdot (-2) \cdot 25 \). Hence \( a = \frac{1}{5} \) and \( b = \frac{2}{25} \).

(c) \( f(x) = a(x - 2)^3 \) or \( f(x) = a(x - 2)Q(x) \) where \( Q(x) \) is a quadratic polynomial without zero.

(d) By intermediate value theorem, there is no cubic polynomial with no zero.

2.
We have cubic function $f(x) = x(x + 2)(x - 2) = x^3 - 4x$. The derivative is $f'(x) = 3x^2 - 4$, which is zero when $x = \pm \frac{2}{\sqrt{3}}$. Hence the points with horizontal tangent lines are $x = \pm \frac{2}{\sqrt{3}}$.

3. The cubic polynomial has local extremum at $x = 2$ and $x = 5$ means that the derivative has zeros at $x = 2, 5$. Hence $f'(x) = a(x - 2)(x - 5)$. We know $a$ should be negative because $f(x)$ is increasing between $[2, 5]$. Then $f(x)$, as the antiderivative of $f'(x) = a(x^2 - 7x + 10)$, is $a\left(\frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x\right)$, where $a$ is negative.

![Graph of a cubic function](image1)

4. (a) $P(x) = a(x - 1)^2(x - 5)(x - e)$. Since $P(0) = 7$, we have $7 = a \cdot 5e$ and $a = \frac{7}{5e}$.

![Graph of a degree 5 polynomial](image2)

(b) A degree 5 polynomial must have a zero.

(c) We can have $P(x) = -x^6 - 10$. 
5. (a) A degree $n$ polynomial has at most $n$ zeros. If $n$ is odd, then it has at least one zero. If $n$ is even, it may happen that it has no zeros at all.

(b) Points with horizontal tangent lines correspond to zeros of the derivative, which is a degree $n - 1$ polynomial. Hence there can be at most $n - 1$ points with horizontal tangent lines. If $n$ is even, there is at least one such point. If $n$ is odd, there can be no such point at all.