1. (a) How would you use what we have learned so far to approximate $\sqrt{3}$? Is your approximation an overestimate or underestimate?

(b) Now try to approximate $\sqrt[3]{9}$ using both methods. Are they overestimates or underestimates?
2. (a) Review how we use the limit definition to obtain the derivative of $\sqrt{x}$.

(b) Can you think of a way to use the product rule $(fg)' = f'g + fg'$ instead to derive $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$? (This proves the power rule for $(x^n)' = nx^{n-1}$ for $n = \frac{1}{2}$.)

(c) Use the limit definition to find the derivative of $\sqrt{x}$.

(d) Use the product rule instead to find the derivative of $\sqrt{x}$. 
Linear Approximation—Solutions

1. (a) **Tangent line method:**
   We choose to make use of the tangent line at (4, 2) to approximate, as we know the square root of 4 and it is near 3. By power rule, if \( f(x) = \sqrt{x} \), then \( f'(x) = \frac{1}{2\sqrt{x}} \), and thus \( f'(4) = \frac{1}{4} \). Using point-slope formula, we find the equation of the tangent line to \( f \) at (4, 2):
   \[
y - 2 = \frac{1}{4}(x - 4)
   \]
   The point on the tangent line with \( x = 3 \) has \( y = \frac{7}{4} \), which is our approximation for \( \sqrt{3} \). This is an underestimate since \( \sqrt{x} \) is a concave down function.

   **Secant line method:**
   We can also use the secant line through the points (1, 1) and (4, 2) to do approximation. The equation of the secant line is
   \[
y - 1 = \frac{1}{3}(x - 1)
   \]
   Hence the approximated value of \( \sqrt{3} \) is \( \frac{5}{3} \). This is an underestimate since \( \sqrt{x} \) is a concave down function and \( x = 3 \) is between the two endpoints \( x = 1 \) and \( x = 4 \) of the secant line.

(b) **Tangent line method:**
   We will use the tangent line at (8, 2). Let \( f(x) = \sqrt{x} \). Compute \( f'(x) = \frac{1}{3\sqrt{x^2}} \) and thus \( f'(8) = \frac{1}{12} \). So the equation of the tangent line is
   \[
y - 2 = \frac{1}{12}(x - 8)
   \]
   and the approximated value of \( \sqrt{9} \) is \( \frac{25}{12} \). This is an underestimate.

   **Secant line method:**
   We use the secant line through (1, 1) and (8, 2). The equation of the secant line is
   \[
y - 1 = \frac{1}{7}(x - 1)
   \]
   so the approximated value of \( \sqrt{9} \) is \( \frac{15}{7} \). This is an overestimate, too, since \( \sqrt{x} \) is concave up and \( x = 9 \) is outside of the two endpoints \( x = 1 \) and \( x = 8 \) of the secant line.
2. (a) 

\[
\left(\sqrt[3]{x}\right)' = \lim_{y \to x} \frac{\sqrt[3]{y} - \sqrt[3]{x}}{y - x} = \lim_{y \to x} \frac{(\sqrt[3]{y} - \sqrt[3]{x})(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})}{(y - x)(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})} = \lim_{y \to x} \frac{y - x}{(y - x)(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})} = \frac{1}{2\sqrt[3]{x}}
\]

(b) Let \( f(x) = \sqrt{x} \). Then \( f(x)^2 = x \). Take the derivative of both sides. The derivative of the right hand side is 1. The derivative of the left hand side can be found by using the product rule:

\[
[f(x)^2]' = [f(x) \cdot f(x)]' = f'(x) \cdot f(x) + f(x) \cdot f'(x) = 2f(x)f'(x)
\]

We have \( 2f'(x)f(x) = 1 \) and thus \( f'(x) = \frac{1}{2f(x)} = \frac{1}{2\sqrt{x}} \).

(c)

\[
\left(\sqrt[3]{x}\right)' = \lim_{y \to x} \frac{\sqrt[3]{y} - \sqrt[3]{x}}{y - x} = \lim_{y \to x} \frac{(\sqrt[3]{y} - \sqrt[3]{x})(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})}{(y - x)(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})} = \lim_{y \to x} \frac{y - x}{(y - x)(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})} = \lim_{y \to x} \frac{1}{(\sqrt[3]{y^2} + \sqrt[3]{y^2} \sqrt[3]{x} + \sqrt[3]{x^2})} = \frac{1}{3\sqrt[3]{x^2}}
\]

(d) Let \( f(x) = \sqrt[3]{x} \). Then \( f(x) \cdot f(x) \cdot f(x) = x \). Take the derivative of both sides. The derivative of the right hand side is 1. The derivative of the left hand side can be found by using the product rule:

\[
f(x)^3)' = [f(x)^2 \cdot f(x)]'
= [f(x)^2]' \cdot f(x) + f(x)^2 \cdot [f(x)]'
= [2f(x)f'(x)] \cdot f(x) + f(x)^2 \cdot f'(x)
= 3f(x)^2f'(x)
\]

We have \( 3f'(x)f(x) = 1 \) and thus \( f'(x) = \frac{1}{3f(x)} = \frac{1}{3\sqrt[3]{x}} \).