1. Let \( f(x) = x^2 \).

   (a) Carefully write down each step of the calculation of \( f'(1) \) from definition.

   (b) Sketch the graph of the function \( \frac{f(x) - f(1)}{x - 1} \).

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**Definition**

We write

\[
\text{the number which } f(x) \text{ approaches as } x \text{ gets really close, but not equal to, } a.
\]

... to mean “the number which \( f(x) \) approaches as \( x \) gets really close, but not equal to, \( a \). It is the limit of \( f(x) \) as \( x \) approaches \( a \).
2. What is $\lim_{x \to 0} f(x)$ in each of the following cases? How about $f(0)$?

(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

(h)
3. (Food for thought)

Let

\[ f(x) = \begin{cases} 
1 & \text{if } x = \pm \frac{1}{n} \text{ for some integer } n \\
0 & \text{otherwise}
\end{cases} \]

be a function defined for all real numbers. What is \( \lim_{x \to 0} f(x) \)?

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**Definition**

- The limit of \( f(x) \) as \( x \) approaches infinity

is the number that \( f(x) \) approaches as \( x \) gets arbitrarily large.

- The limit of \( f(x) \) as \( x \) approaches negative infinity

is the number that \( f(x) \) approaches as \( x \) gets arbitrarily negative.

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4. What is \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) in each of the following cases?

(a) ![Graph](image1)

(b) ![Graph](image2)
Definition

- The limit of $f(x)$ as $x$ approaches $a$ from the left,

is the number that $f(x)$ approaches as $x$ gets really close to $a$, while remaining slightly less than $a$.

- The limit of $f(x)$ as $x$ approaches $a$ from the right,

is the number that $f(x)$ approaches as $x$ gets really close to $a$, while remaining slightly greater than $a$.

5. (a) Sketch the graph of $|x|$ and $\frac{|x|}{x}$.

Determine the followings

(b) $\lim_{x \to 0^+} \frac{|x|}{x}$  
(c) $\lim_{x \to 0^-} \frac{|x|}{x}$  
(d) $\lim_{x \to 0} \frac{|x|}{x}$

(e) What is the derivative of $f(x) = |x|$?
Limits – Solutions

1. (a) 
\[ f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1}(x + 1) = 2 \]

(b) 

2. (a) \( \lim_{x \to 0} f(x) = 2 \), \( f(0) \) does not exist.
   (b) \( \lim_{x \to 0} f(x) = 2 \), \( f(0) = 0 \).
   (c) \( \lim_{x \to 0} f(x) = 1 \), \( f(0) = 1 \).
   (d) \( \lim_{x \to 0} f(x) \) does not exist, \( f(0) = 1.5 \).
   (e) \( \lim_{x \to 0} f(x) \) does not exist, \( f(0) \) does not exist.
   (f) \( \lim_{x \to 0} f(x) = 0 \), \( f(0) \) does not exist.
   (g) \( \lim_{x \to 0} f(x) \) does not exist (with the type of infinity), \( f(0) \) does not exist.
   (h) \( \lim_{x \to 0} f(x) \) does not exist, \( f(0) \) does not exist.

3. \( \lim_{x \to 0} f(x) \) does not exist. The reason is that no matter how close \( x \) gets to 0, if we get a little bit more closer to some \( x = \frac{1}{2^n} \), \( f(x) \) jumps away from 0 to 1. Here is an illustrating picture
4. (a) \( \lim_{x \to \infty} f(x) = 0, \lim_{x \to -\infty} f(x) = 0 \).
(b) \( \lim_{x \to \infty} f(x) = 1, \lim_{x \to -\infty} f(x) \) does not exist.

5. (a) The graph of \( |x| \) is

(b) \( \lim_{x \to 0^+} \frac{|x|}{x} = 1 \).
(c) \( \lim_{x \to 0^-} \frac{|x|}{x} = -1 \).

(d) \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist.

(e) 

\[
    f'(x) = \begin{cases} 
    1 & x > 0 \\
    -1 & x < 0 
\end{cases}
\]