1. Donagh sets off a toy rocket straight up into the air. The function $f(t) = t^2$ describes the toy rocket’s height in meters $t$ seconds after liftoff.

(a) Fill in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>average speed between 3 and $t$ seconds after liftoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td></td>
</tr>
</tbody>
</table>

(b) Fill in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>average speed between 3 and $t$ seconds after liftoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td></td>
</tr>
</tbody>
</table>

(c) What do you think the (instantaneous) speed of the toy rocket 3 seconds after liftoff should be?

(d) Sketch a graph describing the toy rocket’s height $t$ seconds after liftoff.

(e) Interpret what you did in (a)–(c) in terms of the graph.
Definition

Suppose $f(x)$ is a function, $a$ is a number in the domain of $f$. The derivative of $f$ at $a$, written as $f'(a)$, and read as “$f$ prime of $a$”, is

$$f'(a) = \quad \text{or} \quad$$

if it exists.

- $f'(a)$ is the instantaneous rate of change of $f$ at $x = a$.
- Graphically, $f'(a)$ is the slope of the tangent line to the graph $y = f(x)$ at $x = a$.

2. Let $f(x) = \frac{1}{x}$. **If you have more time, do the same problems again for $g(x) = \frac{1}{4x}$.**

   (a) Calculate $f'(1)$ using the definition of the derivative $f$ at $1$.

   (b) Calculate the average rate of change of $f$ between

   - $x = 1$ and $x = 1.5$    
   - $x = 0.5$ and $x = 1$    
   - $x = 0.75, 1.25$

   Do the answers justify your answer in (a)?

   (c) Sketch the graph of $f$ and the tangent line to the graph of $y = f(x)$ at $x = 1$. Is it consistent with your answer in (a)?
(d) Find the equation of the tangent line to the graph of \( y = f(x) \) at \( x = 1 \).

(e) Use the tangent line to approximate \( \frac{1}{1.1} \). Is it an overestimate or an underestimate?

(f) Use the tangent line to approximate \( \frac{1}{0.9} \). Is it an overestimate or an underestimate?

3. (a) Let \( A(r) \) be the area of a circle of radius \( r \) cm. Find the derivative \( A'(5) \). What is the unit of \( A'(5) \)? Explain the meaning of \( A'(5) \) in words.

(b) Let \( B(x) \) be the area of a square with side length \( x \) cm. Find the derivative \( B'(5) \). What is the unit of \( B'(5) \)? Explain the meaning of \( B'(5) \) in words.

Explanation in Pictures:
Observation

- For a concave up graph, a tangent line is ________ the graph, while a secant line is ________ the graph between the endpoints of the secant line, and ________ the graph outside the endpoints.
- For a concave down graph, a tangent line is ________ the graph, while a secant line is ________ the graph between the endpoints of the secant line, and ________ the graph outside the endpoints.

4. Peter is roasting a 14 lb turkey as a test run for Thanksgiving dinner. He begins at noon. At 1:00pm, he checks on the temperature and discovers that it has an internal reading of 35.6°C,¹ and it’s rising at an instantaneous rate of 0.25°C per minute.

(a) Approximate the temperature of the turkey at 1:06pm.

Solution. The temperature of the turkey at 1:00 pm is 35.6°C. If we assume that the temperature continues rises at 0.25°C per minute for the next 6 minutes, then the turkey’s temperature will be 35.6 + 6 \cdot 0.25 = 37.1°C at 1:06 pm.

(b) Let \( I(t) \) be the turkey’s internal temperature (in °C) \( t \) minutes after noon. Use functional notation to express what you were told about the turkey.

Solution. We were told that \( I(60) = 35.6 \) and \( I'(60) = 0.25 \).

(c) Here is a graph of \( I(t) \). Use a sketch to explain the approximation you made in (a).

![Graph of I(t)](image)

Solution. In 0a, we assumed that the turkey’s temperature would continue to rise.

¹According to the USDA, a turkey should be roasted to an internal temperature of 73°C.
rise at the rate of 0.25°C for the next 6 minutes; that is, we assumed that the
turkey’s temperature would change linearly.

(d) Based on your sketch, was your approximation too high or too low?

(e) Would you be comfortable using the same method to predict the turkey’s temper-
ature at 3 pm? Explain. Could we use the definition of the derivative to get
the exact answer in this case?
Definition of the Derivative – Solutions

1. (a) 

<table>
<thead>
<tr>
<th>$t$</th>
<th>average speed between 3 and $t$ seconds after liftoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3.5</td>
<td>6.5</td>
</tr>
<tr>
<td>3.1</td>
<td>6.1</td>
</tr>
</tbody>
</table>

(b) 

<table>
<thead>
<tr>
<th>$t$</th>
<th>average speed between $t$ and 3 seconds after liftoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2.5</td>
<td>5.5</td>
</tr>
<tr>
<td>2.9</td>
<td>5.9</td>
</tr>
</tbody>
</table>

(c) From the two tables above, it seems that the average speed between $t$ seconds and 3 seconds gets closer and closer to 6 as $t$ gets closer to 3. Hence it would be fair to guess that the instantaneous speed at 3 is 6.

(d) 

(e) What we were doing in (a)(b) was computing the slope of the secant line passing through $t$ and 3, and let the point with coordinate $t$ goes closer and closer to the point with coordinate 3. Meanwhile the secant line through the two points will become the tangent line at $t = 3$. 

\[
\begin{array}{c|c}
\hline
f(t) & \hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
& 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
f(t) & 10 & 20 & 30 & \ldots & \ldots & \ldots \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
& t \\
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\end{array}
\]
\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

2. (a) 
\[ f'(a) = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} = \lim_{x \to 1} \frac{1 - x}{x(x - 1)} = \lim_{x \to 1} \frac{-1}{x} = -1 \]

or
\[ f'(a) = \lim_{h \to 0} \frac{1/h - 1}{h} = \lim_{h \to 0} \frac{1 - (1 + h)}{h(1 + h)} = \lim_{h \to 0} \frac{-h}{h(1 + h)} = \lim_{h \to 0} \frac{-1}{1 + h} = -1 \]

(b)  
- The average rate of change of \( f \) in \([1, 1.5]\) is \( \frac{1.5 - 1}{1.5 - 1} = \frac{\frac{1}{3} - 1}{\frac{1}{3} - 1} = -\frac{2}{3} \).
- The average rate of change of \( f \) in \([0.5, 1]\) is \( \frac{1 - 0.5}{1 - 0.5} = \frac{-1}{0.5} = -2 \).
- The average rate of change of \( f \) in \([0.75, 1.25]\) is \( \frac{1.25 - 0.75}{1.25 - 0.75} = \frac{\frac{5}{4} - \frac{3}{4}}{\frac{5}{4} - \frac{3}{4}} = -1 \).

All of the above are not too far from the instantaneous rate of change \(-1\) we get in (a).

(c) 

(d) The tangent line of \( y = f(x) \) at \( x = 1 \) has slope \(-1\), as shown in (a). Using point-slope formula, the equation of the tangent line is \( y - 1 = -(x - 1) \), or \( y = -x + 2 \).

(e) \( \frac{1}{1.1} \approx -1.1 + 2 = 0.9 \). As \( \frac{1}{1.1} = \frac{10}{11} > \frac{9}{10} = 0.9 \), this is an underestimate. That this is an underestimate can also be seen from the graph, as \( y = \frac{1}{x} \) is concave up around \( x = 1 \).
(f) \( \frac{1}{0.9} \approx -0.9 + 2 = 1.1 \). As \( \frac{1}{0.9} = \frac{10}{9} > \frac{11}{10} = 1.1 \), this is an underestimate. That this is an underestimate can also be seen from the graph, as \( y = \frac{1}{x} \) is concave up around \( x = 1 \).

3. (a) The area of a circle with radius \( r \) is \( A(r) = \pi r^2 \). Then by definition of derivative of \( A \) at 5

\[
A'(5) = \lim_{r \to 5} \frac{\pi r^2 - 25\pi}{r - 5} = \lim_{r \to 5} \frac{\pi(r + 5)(r - 5)}{r - 5} = \lim_{r \to 5} \pi(r + 5) = 10\pi
\]

The unit of \( A'(5) \) is \( \text{cm}^2/\text{cm} \). This means that when the circle has radius 5, if we increase the radius by 1 cm, the area increases roughly by \( 10\pi \) cm\(^2\).

(b) The area of a square with side length \( x \) is \( B(x) = x^2 \). Then by definition of derivative of \( B \) at 5

\[
B'(5) = \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x + 5)(x - 5)}{x - 5} = \lim_{x \to 5} (x + 5) = 10
\]

The unit of \( B'(5) \) is \( \text{cm}^2/\text{cm} \). This means that when the square has side length 5, if we increase the side length by 1 cm, the area increases roughly by 10 cm\(^2\).

We observe that \( A'(5) = 10\pi \) is exactly the circumference of the circle of radius 5.