Applications of Hilbert Functions of Zero Dimensional Varieties and Schemes

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26 July 2006

1 Introduction

A problem long studied in mathematics is that of describing, for a given projective variety or projective scheme $X \subset \mathbb{P}^n$, the polynomial functions that vanish on X, or in other words the hypersurfaces that contain X. An important approach to this problem is to determine, for a given m, the number of independent hypersurfaces of degree m

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*Report is submitted towards fulfillment of minor thesis requirement.

containing X. This amounts to determining the dimension of the vector space V of homogeneous polynomials of degree m vanishing on X.

A useful way of expressing this information is in terms of the codimension of V in the vector space of all homogeneous polynomials of degree m on $\mathbb{P}^n$. Hence we define the *Hilbert function* of $X \subset \mathbb{P}^n$, $h_X : \mathbb{N} \to \mathbb{N}$, by taking

$$h_X(m) = \dim S(X)_m \ ,$$

where $S(X)_m$ denotes the mth graded piece of the homogeneous coordinate ring $S(X) = K[Z_0, \dots, Z_m]/I(X)$.

Studying the Hilbert functions of zero dimensional projective varieties and schemes is the basis for many applications in mathematics and applied mathematics. In this paper, we shall summarize some examples and results concerning zero dimensional varieties and schemes, where the given varieties or schemes Γ fit into two main categories:

i. $\Gamma = H \cap C$, where $H \subset \mathbb{P}^n$ is a hyperplane and $C \subset \mathbb{P}^n$ is a reduced, irreducible, and non-degenerate curve.

ii. $\Gamma = H_1 \cap \cdots \cap H_n$, where the $H_i \subset \mathbb{P}^n$ are n distinct hypersurfaces of given degrees d_i meeting transversely.

The collections of points Γ in (i.) figure in Castelnuovo Theory, where, in particular, their Hilbert functions are used to derive Castelnuovo's bound $\pi(d, n)$ for the genus of such a curve C, relative to its degree d and the dimension n of its ambient space. The varieties (or schemes) Γ in (ii.) form the basis for Cayley-Bacharach Theory.

2 Notation and Initial Examples

Throughout this paper we will denote collections of points in projective space by uppercase Greek letters, and the number of distinct points in each collection by the corresponding lowercase Greek letter. So, for instance, we will denote by Γ a collection of γ distinct points $p_1, \ldots, p_\gamma \in \mathbb{P}^n$. We will also use the notation

$$\overline{h}_X(m) = h_X(m) - h_X(m-1) \ .$$

to denote the successive differences of the Hilbert function on a scheme X.

Note that for a given point $p \in \mathbb{P}^n$, the condition that a homogeneous polynomial F vanishes at p is a nontrivial linear condition on the coefficients of F. So γ distinct points impose γ linear conditions on F—conditions which are linearly independent when applied to the infinite dimensional space $K[X_0, \ldots, X_n]$ of all homogeneous polynomials over the base field K.

The conditions will in general lose their independence, however, when applied to a finite dimensional subspace of $K[X_0, \ldots, X_n]$, such as the subspace $K[X_0, \ldots, X_n]_m$ of homogeneous polynomials of degree m. Hence the function $h_\Gamma(m)$ counts the number of independent conditions imposed by the points of Γ on hypersurfaces of degree m. Certainly $h_\Gamma(m) \leq \gamma$. Moreover, for any scheme $X \subset \mathbb{P}^n$, we have $h_X(0) = 1$.

To take a basic example, let Γ be three distinct points $p_1, p_2, p_3 \in \mathbb{P}^2$. If the points are collinear, then $h_\Gamma(1) = 2$. Otherwise $h_\Gamma(1) = 3$. In either case $h_\Gamma(2) = 3$, because we can construct a homogeneous quadratic polynomial that vanishes on any two of the points but not on the third. Simply take the product of a linear form that vanishes on the first point (say p_1) but not on the third point (say p_3) with a linear form that vanishes on the second point (p_2) but not on the third point (p_3).

Similarly, if Γ consists of four distinct points in the projective plane, then $h_\Gamma(1) = 2$ if the points are collinear, and $h_\Gamma(1) = 3$ if they are not. If Γ was the collection any four distinct points in $\mathbb{P}^3$, then $h_\Gamma(1) = 3$ if the points are coplanar, and $h_\Gamma(1) = 4$ otherwise. In fact, it is true for any collection of points $\Gamma = \{p_1, \ldots, p_\gamma\} \subset \mathbb{P}^n$ that we have

$$h_\Gamma(1) = \dim \overline{p_1, \ldots, p_\gamma} + 1 \ ,$$

which is the dimension of the linear subspace of K^{n+1} spanned by the one dimensional subspaces of K^{n+1} corresponding to the points of Γ.

Observe that for a given Γ and for every $m \geq 0$, we have

$$h_\Gamma(m) \ \leq \ h_\Gamma(m+1) \ .$$

This is because we can find a homogeneous polynomial F of degree m that vanishes on $h_\Gamma(m) - 1$ points of Γ but not on any one of the remaining points, say p_γ. We can then multiply F by any linear form not vanishing on p_γ to get a homogeneous polynomial of degree $m+1$ vanishing on $h_\Gamma(m) - 1$ points of Γ but not on all of Γ. Hence for any collection of points Γ, the function h_Γ is positive and increasing.

Furthermore, we can see that for any given $\Gamma = \{p_1, \ldots, p_\gamma\}$ and for any $m \geq \gamma - 1$, we have $h_\Gamma(m) = \gamma$. This follows from the previous observation and because we can construct a polynomial of degree $\gamma - 1$ vanishing on the points $p_1, \ldots, p_{\gamma-1}$ but not on p_γ. We simply take the product of $\gamma - 1$ linear forms L_i where each L_i vanishes on p_i but not on p_γ.

The following proposition summarizes what we have observed so far:

Proposition 1. *For a given collection of points* $\Gamma = \{p_1, \ldots, p_\gamma\} \subset \mathbb{P}^n$*, we have*

i. $h_\Gamma(0) = 1$,

ii. $h_\Gamma(1) = dim\ \overline{p_1, \ldots, p_\gamma} + 1$,

iii. $h_\Gamma(m) \leq h_\Gamma(m+1) \leq \gamma$ *for every* $m \geq 0$, *and* *iv.* $h_\Gamma(m) = \gamma$ *for every* $m \geq \gamma - 1$.

Proof. See above. $\square$

3 Castelnuovo Theory

In this section we will consider collections of points $\Gamma = H \cap C$, where $H \subset \mathbb{P}^n$ is a hyperplane and $C \subset \mathbb{P}^n$ is a reduced, irreducible, and non-degenerate curve. We will use the following result:

Lemma 2. (Uniform Position) (Harris) *For any reduced, irreducible, and non-degenerate curve* $C \subset \mathbb{P}^n$ *of degree* d*, let* Γ *be its general hyperplane section. Then for any subset* $\Omega \subset \Gamma$ *with* ω *points, and for all* m*, we have*

$$h_\Omega(m) = min(\omega, h_\Gamma(m)) \ .$$

Proof. Writing $\Gamma = H \cap C$, we claim that the monodromy group M of Γ as H varies is twice transitive and contains a transposition—hence is the full symmetric group. For any $t \geq 1$, we set $I_t \subset \mathbb{P}^n* \times C^{(t)}$ with

$$I_t = \{(H; p_1, \ldots, p_t) : p_1, \ldots, p_t \in H \cap C \text{ are all distinct}\} \ .$$

Now $I_2 \subset \mathbb{P}^n* \times C \times C$ maps onto $C \times C - \Delta$. The fiber of a pair of points (p_1, p_2) is the space of hyperplanes in $\mathbb{P}^n$ containing the line spanned by p_1 and p_2, hence is isomorphic to $\mathbb{P}^{n-2}$. So in particular I_2 is irreducible and therefore connected, giving the twice transitivity of M.

To see that M contains a transposition, consider the dual curve to C, denoted by $C* \subset \mathbb{P}^n*$. Let H_0 be a hyperplane that is simply tangent to C at a point p_0, or in other words $H_0 \in C* - C*_{\text{sing}}$ and $H_0 \in C* \cap p_0*$. Let

$$L = \{H_\lambda : |\lambda| < \epsilon\} \subset \mathbb{P}^n*$$

be a one parameter family of hyperplanes such that L meets C^* transversely at H_0, choosing ϵ small enough that every $H_\lambda \in L$ meets C transversely. Then for every $0 < |\lambda| < \epsilon$ there exist two distinct points $p_\lambda, q_\lambda \in H_\lambda \cap C$ both approaching p_0 as $\lambda \to 0$. Since p_λ and q_λ switch places when λ loops once around zero, M contains the corresponding transposition.

Note that since M is the full symmetric group on d letters, M is t transitive for any $1 \leq t \leq d$, and hence I_t is irreducible.

Now let $J \subset \mathbb{P}^{n*} \times C^{(t_0)}$ with

$$J = \{(H; p_1, \ldots, p_{t_0}) : p_1, \ldots, p_{t_0} \in H \cap C \text{ are dependent}\} ,$$

where $t_0 = h_\Gamma(m)$. We see that J is a proper closed subvariety of I_{t_0}, and since I_{t_0} is irreducible, $\dim J < \dim I_{t_0}$ and so the image of J under projection to $\mathbb{P}^{n*}$ is a proper subvariety of $\mathbb{P}^{n*}$.

Hence there is an open subset of $\mathbb{P}^{n*}$ that avoids the image of J, or in other words, for a general hyperplane $H \in \mathbb{P}^{n*}$ and $\Gamma = H \cap C$ and for a given m, any $h_\Gamma(m)$ points of Γ impose independent conditions on polynomials of degree m.

Hence given any $\Omega \subset \Gamma$ with ω points, one of two cases occurs. Suppose first that $\omega \leq h_\Gamma(m)$ and so the points of Ω impose independent conditions on hypersurfaces of degree m; we therefore have $h_\Omega(m) = \omega$. On the other hand, if $\omega > h_\Gamma(m)$, then we must have $h_\Omega(m) \geq h_\Gamma(m)$ because any $h_\Gamma(m)$ points of Ω are independent, and $h_\Omega(m) \leq h_\Gamma(m)$ because $\Omega \subset \Gamma$, and thus $h_\Omega(m) = h_\Gamma(m)$.

Hence if Ω fails to impose independent conditions on polynomials of degree m, then any hypersurface of degree m containing Ω must contain all of Γ. $\square$

We can now prove the following consequence of Lemma 2.

Corollary 3. (Harris and Eisenbud) *For any reduced, irreducible, and non-degenerate curve $C \subset \mathbb{P}^n$ of degree d, let Γ be its general hyperplane section. Then given any nonnegative integers m and n, we have*

$$h_\Gamma(m+n) \geq min\,(d, h_\Gamma(m) + h_\Gamma(n) - 1) \ .$$

Proof. First suppose that $d \geq h_\Gamma(m) + h_\Gamma(n)$. Let Ω_m be a subset of $h_\Gamma(m)$ points of Γ and Ω_n be a subset of $h_\Gamma(n)$ points of Γ, such that $\Omega_m \cap \Omega_n$ contains exactly one point p. Then for $i = m, n$ there exists a polynomial F_i of degree i vanishing on $\Omega_i - \{p\}$ but not on p. But the product $F_m F_n$ is a polynomial of degree $m+n$ vanishing on $\Omega_m \cap \Omega_n - \{p\}$ but not on p, and so

$$h_\Gamma(m+n) \geq \#(\Omega_m \cap \Omega_n) = h_\Gamma(m) + h_\Gamma(n) - 1 \ .$$

Now suppose that $d < h_\Gamma(m) + h_\Gamma(n)$. Let Ω_m and Ω_n be as above, except let $\Omega_m \cap \Omega_n$ now contain exactly $\delta = h_\Gamma(m) + h_\Gamma(n) - d$ points $p_1, \ldots, p_\delta$, so that $\Omega_m \cap \Omega_n$ is all of Γ. As before, we have for $i = m, n$ a polynomial F_i of degree i vanishing on $\Omega_i - \{p_\delta\}$ but not on p_δ. The product $F_m F_n$ is a polynomial of degree $m+n$ vanishing on $\Gamma - \{p_\delta\}$ but not on p_δ, and so

$$h_\Gamma(m+n) \geq \#(\Gamma) = d \ .$$

□

Applying Corollary 3, we get the further result:

Corollary 4. *Let Γ be a general hyperplane section of a reduced, irreducible, and non-degenerate curve C of degree d in $\mathbb{P}^n$. Then for every $m \geq 1$, either $h_\Gamma(m) = d$ or $\overline{h}_\Gamma(m) \geq n - 1$.*

Proof. Note that if C is non-degenerate, then Γ is non-degenerate; hence $h_\Gamma(1) = n$. By the previous corollary, for every $m \geq 1$ we have

$$h_\Gamma(m) \geq \min\left(d, h_\Gamma(m-1) + n - 1\right).$$

But in any case we have $h_\Gamma(m) \leq d$, and so either $h_\Gamma(m) = d$ or $\overline{h}_\Gamma(m) \geq n - 1$. $\square$

Before applying the foregoing results to derivation of Castelnuovo's bound on the genus of a curve, we must establish the following lemma, which applies to arbitrary schemes and their hyperplane sections.

Lemma 5. (Harris and Eisenbud) *For any scheme $X \subset \mathbb{P}^n$, if Y is a hyperplane section of X not containing any component of X, then*

$$h_Y(m) \leq \overline{h}_X(m)$$

for all $m \geq 1$. Moreover, equality holds if and only if every polynomial in $\mathbb{P}^{n-1}$ vanishing on Y is the restriction of a polynomial in $\mathbb{P}^n$ vanishing on X.

Proof. Let D_m be the linear series cut on X by hypersurfaces of degree m and let $E_m \subset D_m$ be the linear system of divisors containing Y. Now the codimension of E_m in D_m is just $h_Y(m)$. Moreover, the series of divisors $D_{m-1} + Y$ is certainly contained in E_m. So $h_Y(m)$ is at most dim $D_m -$ dim D_{m-1}. But dim $D_m = h_X(m)$, so

$$h_Y(m) \leq h_X(m) - h_X(m-1) = \overline{h}_X(m) .$$

Note further that $h_Y(m) = \overline{h}_X(m)$ if and only if

$$\dim E_m = \dim D_{m-1} ,$$

which is equivalent to $E_m = D_{m-1} + Y$, or, in other words, every polynomial in $\mathbb{P}^{n-1}$ vanishing on Y is the restriction of a polynomial in $\mathbb{P}^n$ vanishing on X. $\square$

In order to relate the genus of a curve to the Hilbert functions of the curve itself and its general hyperplane section, we must observe that for a curve $C \in \mathbb{P}^n$ of degree d and genus g, we have that for large m,

$$h_C(m) = md - g + 1 \ .$$

(This is in fact what is called the *Hilbert polynomial* of C, a polynomial function that equals the Hilbert function of C for large enough values of m.) We get the formula by first choosing coordinates $Z_0, \dots, Z_n$ for $\mathbb{P}^n$ so that the hyperplane $H_0 = (Z_0 = 0)$ intersects C transversely. Write $H_0 \cap C = \{p_1, \dots, p_d\}$.

Next we define a map θ from the mth graded piece of the coordinate ring $S(C) = K[Z_0, \dots, Z_n]/I(C)$ (homogeneous polynomials of degree m vanishing on C) to the space of meromorphic functions with poles (of order at most m) only at the points of $H_0 \cap C$, by

$$\begin{aligned} \theta : S(C)_m &\rightarrow \mathscr{L}(m \cdot p_1 + \cdots + m \cdot p_d) \ ; \\ \theta\,(F(Z)) &= F(Z)/Z_0^m \ . \end{aligned}$$

For large enough m, θ is an isomorphism and the divisor $D = m \cdot p_1 + \cdots + m \cdot p_d$ on C is nonspecial—that is, $\dim \mathscr{L}(K_C - D) = 0$. So for large m, we have $h_C(m) = \dim \mathscr{L}(m \cdot p_1 + \cdots + m \cdot p_d)$ and by Riemann-Roch,

$$h_C(m) = \dim \mathscr{L}(m \cdot p_1 + \cdots + m \cdot p_d) = md - g + 1 \ ,$$

which is what we need to prove the following corollary to Lemma 5.

Corollary 6. *Let Γ be a hyperplane section of a curve $C \subset \mathbb{P}^n$ of degree d and genus g. If Γ does not contain any component of C, then*

$$g \leq \sum_{k=1}^{\infty} d - h_\Gamma(k) = \sum_{k=1}^{\infty} (k-1) \cdot \overline{h}_\Gamma(k) \ .$$

Proof. For large m, we have

$$\begin{aligned} md - g + 1 &= h_C(m) \\ &= 1 + \sum_{k=1}^{m} \overline{h}_C(k) \\ &\geq 1 + \sum_{k=1}^{m} h_\Gamma(k) \end{aligned}$$

and so

$$\begin{aligned} g \leq md - \sum_{k=1}^{m} h_\Gamma(k) &= \sum_{k=1}^{m} d - h_\Gamma(k) \\ &= \sum_{k=1}^{\infty} d - h_\Gamma(k) \\ &= \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \overline{h}_\Gamma(j) \\ &= \sum_{j=2}^{\infty} (j-1) \cdot \overline{h}_\Gamma(j)) \\ &= \sum_{k=1}^{\infty} (k-1) \cdot \overline{h}_\Gamma(k)) \end{aligned}$$

because when m is large enough relative to d, that is if $m \geq d-1$ (see Proposition 1), then for all $k \geq m$, we have $d - h_\Gamma(k) = 0$. $\square$

By combining Corollaries 4 and 6, we can now derive the first part of Castelnuovo's Theorem.

Theorem 7. (Castenuovo's Bound) *Let $C \subset \mathbb{P}^n$ be a reduced, irreducible, and non-degenerate curve of degree d and genus g with general hyperplane section Γ. Write $m = \lfloor \frac{d-1}{n-1} \rfloor$ and $d = m(n-1)+\epsilon+1$ where $0 \leq \epsilon \leq n-2$. Then*

$$g \leq \pi(d,n) = \binom{m}{2}(n-1) + m\epsilon \ .$$

Moreover, equality holds if and only if every polynomial in $\mathbb{P}^{n-1}$ vanishing on Γ is the restriction of a polynomial in $\mathbb{P}^n$ vanishing on C (that is, C is "arithmetically Cohen-Macaulay") and the Hilbert functions of Γ and C are given by

$$i.\ \ \overline{h}_\Gamma(k) = \begin{cases} 1 & \textit{if } k = 0 \ , \\ n-1 & \textit{if } 1 \leq k \leq m \ , \\ \epsilon & \textit{if } k = m+1 \ ; \end{cases}$$

$$ii.\ \ h_\Gamma(k) = \begin{cases} 1 & \textit{if } k = 0 \ , \\ 1 + k(n-1) & \textit{if } 1 \leq k \leq m \ , \\ d & \textit{if } k = m+1 \ ; \end{cases}$$

$$iii.\ \ h_C(k) = \begin{cases} 1 & \textit{if } k = 0 \ , \\ \binom{k+1}{2}(n-1) + k + 1 & \textit{if } 1 \leq k \leq m \ , \\ \binom{m+1}{2}(n-1) + m + 1 + d & \textit{if } k = m+1 \ . \end{cases}$$

Proof. The Hilbert function for Γ given in (i.) and (ii.) above maximizes the upper bound for the genus of C given in Corollary 6, subject to the constraints noted in Corollary 4. Substituting into the inequality of Corollary 6, we get

$$\begin{aligned} g &\leq \sum_{k=1}^{\infty}(k-1)\cdot \overline{h}_{\Gamma}(k) \\ &\leq [(m+1)-1]\cdot\epsilon + \sum_{k=1}^{m}(k-1)(n-1) \\ &= \binom{m}{2}(n-1)+m\epsilon\ . \end{aligned}$$

If equality holds, then the Hilbert function of Γ must be as given, and further more we have

$$\begin{aligned} h_C(m+1) &= (m+1)d-g+1 \\ &= d+m\,[m(n-1)+\epsilon+1]-\left[\binom{m}{2}(n-1)+m\epsilon\right]+1 \\ &= \binom{m+1}{2}(n-1)+m+1+d \end{aligned}$$

and so $\overline{h}_C(k)=h_{\Gamma}(k)$ must hold for each $k\geq 1$ and the Hilbert function of C must be as given in (iii.).

Conversely, if $\overline{h}_C(k)=h_{\Gamma}(k)$ holds for every $k\geq 1$ and the Hilbert function of C is as given in (iii.), then $g=\pi(d,n)$. □

Castelnuovo was able to further characterize the configuration of collections of points Γ whose Hilbert functions are given by *i.* and *ii.* by showing that such a $\Gamma\subset H\cong\mathbb{P}^{n-1}$ with minimal Hilbert function must lie on a rational normal curve, or in other words a non-degenerate, irreducible curve B of degree $n-1$ in $\mathbb{P}^{n-1}$.

Indeed if we take any collection $\Gamma$of $d \geq k(n-1)+1$ points on such a curve B, then by Bézout's Theorem, for each positive integer $j \leq k$ we must have that every hypersurface in $\mathbb{P}^n$ containing Γ must contain B, and it will contain B exactly when it contains any $j(n-1)+1$ points of B, and so

$$h_\Gamma(j) = h_B(j) = j(n-1)+1 \ ,$$

as desired. Castelnuovo showed that, conversely, these are the only examples of Γ with minimal Hilbert functions.

Lemma 8. (Castelnuovo's Lemma) *Suppose $\Gamma \subset \mathbb{P}^n$ is any collection of $\gamma \geq 2n+3$ points in linear general position. If $h_\Gamma(2) = 2n+1$, then Γ lies on a rational normal curve.*

Proof. Now for any subset Ω of $2n+1$ points of Γ, we have $h_\Omega(2) = 2n+1$ because the points are in general position. Hence any quadric containing Ω will contain all of Γ.

For each $1 \leq t \leq n$ let $H_t(\lambda)$ be the pencil of hyperplanes through the points $p_1, \ldots, \hat{p_t}, \ldots, p_n$ and let $H(\lambda)$ be the pencil of hyperplanes through the points $p_{n+1}, \ldots, p_{2n-1}$, such that for each t, we have

$$\begin{aligned} p_{2n} &\in H_t(0), H(0) \ ; \\ p_{2n+1} &\in H_t(1), H(1) \ ; \text{ and} \\ p_{2n+2} &\in H_t(\infty), H(\infty) \ . \end{aligned}$$

We can then find parameters λ_j and $\theta_{t,j}$ such that for $1 \leq j \leq n$, we have $p_j \in H(\lambda_j)$ and for $n+1 \leq j \leq 2n-1$ and for each $1 \leq t \leq n$, we have $p_j \in H_t(\theta_{t,j})$.

Consider the quadrics

$$Q_t = \bigcup_\lambda H_t(\lambda) \cap H(\lambda) \ .$$

For each t, we have $p_1, \ldots, \hat{p}_t, \ldots, p_n, \ldots, p_{2n+2} \in Q_t$, and so we must also have $p_{2n+3}, \ldots, p_\gamma \in Q_t$, or in other words there exist parameters λ_j with $2n+3 \leq j \leq \gamma$ such that for each $2n+3 \leq j \leq \gamma$ and each $1 \leq t \leq n$, we have $p_j \in H_t(\lambda_j) \cap H(\lambda_j)$. Therefore the points $p_1, \ldots, p_n$ and $p_{2n}, \ldots, p_\gamma$ all lie on the rational normal curve

$$B = \bigcup_\lambda H_1(\lambda) \cap \cdots \cap H_n(\lambda) \ .$$

The labeling of the points being arbitrary, we have actually shown that any $\gamma - n + 1 \geq n + 4$ points of Γ lie on a rational normal curve. But any $n+3$ points determine a rational normal curve, and so all the points of Γ lie on a rational normal curve. $\square$

From Castelnuovo's Lemma, one can further deduce that for $d \geq 2n+1$, a curve $C \subset \mathbb{P}^n$ of maximal genus $\pi(d,n)$ with respect to its degree d (called an *extremal* curve) lies on a rational normal scroll (except perhaps in the case $n = 5$, when it may lie on the Veronese surface).

This follows from the observation that since the hyperplane section Γ of C lies on a rational normal curve $B \subset \mathbb{P}^{n-1}$ by Lemma 8 and since $d = \deg \Gamma > 2(n-1)$, every quadric containing Γ must contain B. But B is the intersection of the quadrics in $\mathbb{P}^{n-1}$ containing B, and so B the intersection of the quadrics in $\mathbb{P}^{n-1}$ containing Γ. Since C is arithmetically Cohen-Macaulay, we can moreover deduce that B is the intersection of the quadrics in $\mathbb{P}^n$ containing C and the hyperplane containing Γ. Hence the intersection of the quadrics containing C is a surface S with hyperplane section B. S can be shown to be a rational normal scroll, or, in the case $n = 5$, possibly a cone over the Veronese surface.

The second half of Castelnuovo's original Theorem gives the possible classes of the extremal curve C if the scroll is smooth. We shall not give this result or any further results here, apart from mentioning briefly one generalization; one can find more information on generalizations of Castelnuovo's Theorem and related conjectures in [3] and [4].

Theorem 9. (Eisenbud and Harris) *Set*

$$\begin{aligned} m_1 &= \lfloor \frac{d-1}{n} \rfloor \ , \\ \epsilon_1 &= d - m_1 n - 1 \ , \\ \mu_1 &= \begin{cases} 1 & \textit{if } \epsilon_1 = n-1 \ , \\ 0 & \textit{if } \epsilon_1 \neq n-1 \ , \end{cases} \quad \textit{and} \\ \pi_1(d,n) &= \binom{m_1}{2} n + m_1(\epsilon_1 + 1) + \mu_1 \ . \end{aligned}$$

Then any reduced, irreducible, non-degenerate curve $C \subset \mathbb{P}^n$ *of degree* $d \geq 2n+3$ *and genus* $g > \pi_1(d,n)$ *lies on a surface of degree* $n-1$*.*

For a proof of Theorem 9, see [4].

4 Cayley-Bacharach Theory

We now turn to the second type of zero dimensional variety $\Gamma \subset \mathbb{P}^n$ whose Hilbert functions have been widely studied, and of which the hyperplane sections of curves are a special case. A famous result concerning the Hilbert functions of collections of points $\Gamma = H_1 \cap \cdots \cap H_n$, where the $H_i \subset \mathbb{P}^n$ are n distinct hypersurfaces of given degrees d_i meeting transversely, is the Cayley-Bacharach Theorem. Various versions of this theorem appeared long before Cayley and Bacharach or before Hilbert functions were even formally defined. We shall briefly mention one precursor and then prove the theorem itself.

Note that in the projective plane $\mathbb{P}^2$, a complete intersection of hypersurfaces is just the intersection of two plane curves C_1 and C_2 of degrees d_1 and d_2, respectively. Taking $d_1 = d_2 = 3$, we get:

Theorem 10. (Chasles' Theorem) *Let $C_1, C_2 \subset \mathbb{P}^2$ be cubic curves meeting in nine points $p_1, \ldots, p_9$. If $X \subset \mathbb{P}^2$ is any cubic such that $p_1, \ldots, p_8 \in X$, then moreover $p_9 \in X$.*

From the viewpoint of Hilbert functions, Theorem 10 actually says that if $\Gamma = C_1 \cap C_2 = \{p_1, \ldots, p_9\}$, then we have $h_\Gamma(3) \leq 8$. In fact we can prove the stronger statement that $h_\Gamma(3) = 8$ by showing that any eights points of $C_1 \cap C_2$ impose independent conditions on cubics. This follows from the next proposition.

Proposition 11. *Let $\Omega = \{p_1, \ldots, p_\omega\} \subset \mathbb{P}^2$ be a collection of $\omega \leq 2k + 2$ distinct points. Then $h_\Omega(k) < \omega$ if and only if either $k + 2$ of the points are collinear, or $\gamma = 2k + 2$ and Ω is contained in a conic.*

Proof of the proposition. Suppose first that $k + 2$ of the points of Ω lie on a line L. Then by Bézout, any curve of degree k that contains Ω must also contain L. The codimension of the space of curves containing L is $\binom{k+2}{2} - \binom{k+1}{2} = k + 1$, or in other words the $k + 2$ points of $\Omega \cap L$ impose at most $k + 1$ conditions on curves of degree k, and so all of Ω imposes at most $k + 1 + [\omega - (k + 2)] = \omega - 1$ conditions on curves of degree k.

Similarly, suppose that $\omega = 2k + 2$ and there exists an conic C containing Ω. By Bézout, any curve of degree k containing Ω must contain C, and since the codimension of the space of curves of degree k containing C is $\binom{k+2}{2} - \binom{k}{2} = 2k + 1$, Ω imposes at most $2k + 1$ conditions on curves of degree k.

Conversely, suppose that $h_\Omega(k) < \omega$. By induction on ω, we can assume that for any proper subset $\Lambda \subset \Omega$ of λ points, we have $h_\Lambda(k) =$

λ, so it is enough to show that any curve of degree k containing any $\omega - 1$ points of Ω contains all of Ω.

For the case $k = 1$, note that any set of $\omega \leq 4$ points in the plane fails to impose independent conditions on lines exactly when either $\omega = 3$ and the points are collinear or $\omega = 4$ (and hence the points lie on a pair of double lines).

Note further that for arbitrary k and for any set Ω of $\omega < k + 2$ points, Ω cannot fail to impose independent conditions on curves of degree k, since we have $k \geq \omega - 1$, and so by Proposition 1, $h_{\Omega}(k) = \omega$.

Now suppose that k is arbitrary and $\omega \geq k + 2$. Suppose $h_{\Omega}(k) < \omega$ and assume that there exists a line L containing at least $k + 1$ points of Ω. Set $\Lambda_1 = L \cap \Omega$ and $\Lambda_2 = \Omega - \Lambda_2$, with each $\lambda_t = \#(\Lambda_t)$. Suppose that $\lambda_1 = k + 1$. Then $\lambda_2 = \omega - (k + 1) \leq k + 1$ and we must have $h_{\Lambda_2}(k - 1) < \lambda_2$, as otherwise there would exist a plane curve B of degree $k - 1$ containing all but one point of Λ_2, and so the curve $L \cup B$ of degree k would contain all but one point of Ω. Therefore by induction Λ_2 must contain exactly $k + 1$ points, all of which must lie on a line M. Hence either $\lambda_1 \geq k + 2$ or else Ω is contained in the conic $L \cup M$.

In the next case, assume that there exists a line J containing $j \geq 3$ points of Ω. Then the remaining $\omega - j$ points must fail to impose independent conditions on curves of degree $k - 1$, and hence there exists a line L containing at least $k + 1$ points of Ω, and we are done.

For the final case, assume that no three points of Ω are collinear. Choose points $p_1, p_2, p_3 \in \Omega$ and set $\Lambda_0 = \Omega - \{p_1, p_2, p_3\}$, with $\lambda_0 = \#(\Lambda_0) = \omega - 3$. For $1 \leq t \leq 3$, set $\Lambda_t = \Lambda_0 \cup p_t$. Now for $1 \leq t \leq 3$ we must have $h_{\Lambda_t}(k - 1) < \lambda_0 + 1$, as otherwise there would be a plane curve B of degree $k - 1$ containing Λ_0 but not p_t and so the union of B with the line through the other two points of $\{p_1, p_2, p_3\}$ would be a curve of degree k containing all but one point of Ω. None of the Λ_t can contain $k + 1$ collinear points, and so by induction we must have $\lambda_0 + 1 = 2(k - 1) + 2 = 2k$ (or in other words $\omega = 2k + 2$) and for each $1 \leq t \leq 3$, there exists a conic $C_t \subset \mathbb{P}^2$ such that $\Lambda_t \subset C_t$. Now if $k \geq 3$, then $\lambda_0 \geq 5$ with no three points of Λ_0 collinear and so there can exist at most one conic C containing Λ_0, and hence $C = C_t$ for each $1 \leq t \leq 3$ and so $\Omega \subset C$. Finally, if $k = 2$ then Ω fails to impose independent condition on conics, which can only happen if the six points of Ω lie on a conic. □

Thus we see that in Chasles' Theorem, for any subset Ω of eight distinct points of Γ, we must have $h_\Omega(3) = 8$ because, applying Proposition 11 in the case $k = 3$ and $\omega = 8$ we see that the points of Ω cannot lie on a conic. If they did lie on a conic $Y \subset \mathbb{P}^2$, then since $8 > 2 \cdot 3$, by Bézout both C_1 and C_2 must contain a component of Y, which is impossible because C_1 and C_2 intersect transversely. By the same argument no four of the points of Ω can lie on a line; hence the proposition implies that the eight points $p_1, \ldots, p_8 \in C_1 \cap C_2$ must impose independent conditions on cubics.

We will omit the proof of Theorem 10 here as it will easily follow from the more general result below, proposed by Cayley, corrected by Bacharach, and generalized to $\mathbb{P}^n$:

Theorem 12. (Cayley-Bacharach) *Let* $\Gamma = H_1 \cap \cdots \cap H_n$, *where the* $H_i \subset \mathbb{P}^n$ *are* n *distinct hypersurfaces of given degrees* d_i *meeting transversely. Write* Γ *as the union of disjoint subsets* $\Gamma = \Omega_1 \cup \Omega_2$ *of degrees* ω_1 *and* ω_2, *respectively, and set* $s = -(n+1) + \sum_{i=1}^{n} d_i$. *Then for any* $0 \leq m \leq s$, *we have*

$$h_\Gamma(m) - h_{\Omega_1}(m) = \omega_2 - h_{\Omega_2}(s-m) \ .$$

Theorem 10 is now the application of Theorem 12 in the case $n = 2$, $d_1 = d_2 = 3$, $\Omega_1 = \{p_1, \ldots, p_8\}$, $\Omega_2 = \{p_9\}$, and $m = 3$. Indeed we then have

$$h_\Gamma(3) - 8 = 1 - 1 = 0 \ .$$

We will use without proof the following statement to prove Theorem 12.

Theorem 13. (Restsatz) (Brill and Noether) *For any plane curve* $X \subset \mathbb{P}^2$ *and any given positive integer* k*, the linear series cut on* X *by forms of degree* k *is complete.*

Proof of the Cayley-Bacharach Theorem. Let $X = \bigcap_{i=1}^{n-1} X_i$ and write $d = \prod_{i=1}^{n-1} d_i$. We will prove the theorem only in the case where the plane curve X is nonsingular; by Bertini's and Bézout's Theorems we can reduce to this case.

Let H denote the hyperplane divisor on X. First we observe that by applying the Restsatz Theorem, the linear series $k \cdot H - \Lambda$ on X is complete for any positive integer k and any collection of points $\Lambda \subset X$ (viewed as a divisor on X). Let D be any divisor on X such that $D \sim k \cdot H - \Lambda$. Then $D + \Lambda \sim k \cdot H$, and since the linear series $k \cdot H$ on X is complete, there exists a plane curve C of degree k such that $D + \Lambda = C \cdot X$, and hence $D = C \cdot X - \Lambda$.

Note further that by completeness of the series $k \cdot H$, the dimension of the vector space $\mathscr{L}(k \cdot H)$ is the dimension of the space of forms of degree k modulo those vanishing on X, and similarly by the completeness of $k \cdot H - \Lambda$, the dimension of $\mathscr{L}(k \cdot H - \Lambda)$ is the dimension of the space of forms of degree k vanishing on Λ modulo those vanishing on all of X, or in other words, writing $\ell(D) = \dim \mathscr{L}(D)$, we have

$$h_\Lambda(k) = \ell(k \cdot H) - \ell(k \cdot H - \Lambda) \ .$$

Observing that $\Gamma = d_n \cdot H$ and $\gamma = d_n d$, we see that

$$\begin{aligned} h_\Gamma(m) - h_{\Omega_1}(m) &= [\ell(m \cdot H) - \ell(m \cdot H - \Gamma)] \\ &\quad - [\ell(m \cdot H) - \ell(m \cdot H - \Omega_1)] \\ &= \ell(m \cdot H - \Omega_1) - \ell(m \cdot H - \Gamma) \\ &= \ell(m \cdot H - \Omega_1) - \ell\left([m - d_n] \cdot H\right) \ . \end{aligned}$$

Now, by the adjunction formula, we have

$$K_X \sim \left(-n - 1 + \sum_{i=1}^{n-1} d_i \right) \cdot H = (s - d_n) \cdot H \ ,$$

and so by Riemann-Roch,

$$\begin{aligned} h_\Gamma(m) - h_{\Omega_1}(m) &= \ell(m \cdot H - \Omega_1) - \ell\left([m - d_n] \cdot H\right) \\ &= [md - \omega_1 - g + \ell\left([s - d_n - m] \cdot H + \Omega_1\right)] \\ &\quad - [(m - d_n)d - g + \ell\left([s - d_n - m + d_n] \cdot H\right)] \\ &= d_n d - \omega_1 + \ell\left([s - m] \cdot H - [d_n \cdot H - \Omega_1]\right) \\ &\quad - \ell\left([s - m] \cdot H\right) \\ &= \gamma - \omega_1 + \ell\left([s - m] \cdot H - [\Gamma - \Omega_1]\right) \\ &\quad - \ell\left([s - m] \cdot H\right) \\ &= \omega_2 - [\ell\left([s - m] \cdot H\right) - \ell\left([s - m] \cdot H - \Omega_2\right)] \\ &= \omega_2 - h_{\Omega_2}(s - m) \ , \end{aligned}$$

which completes the proof. □

In recent years there has been much work done on extending the Cayley-Bacharach Theorem, as found in [3] It is beyond the scope of this survey to go into depth on this work, but in closing we will state the modern version of the theorem.

Theorem 14. (Modern Cayley-Bacharach)
(Davis, Geramita, and Orrechia)

Let $\Gamma \subset \mathbb{P}^n$ be a complete intersection of hypersurfaces $X_1, \ldots, X_n$ of degrees $d_1, \ldots, d_n$. Suppose that $\Gamma\prime, \Gamma\prime\prime \subset \Gamma$ are closed subschemes, residual to one another. Write $s = -n - 1 + \sum_{j=1}^{n} d_j$. Then, for any for any $k \geq 0$, the number of hypersurfaces of degree k containing $\Gamma\prime$

modulo those vanishing on Γ *is equal to the failure of* Γ'' *to impose independent conditions on hypersurfaces of degree* $s-k$, *or in other words*

$$h^0\left(\mathbb{P}^n, \mathcal{I}_{\Gamma'}(k)\right) - h^0\left(\mathbb{P}^n, \mathcal{I}_{\Gamma}(k)\right) = h^1\left(\mathbb{P}^n, \mathcal{I}_{\Gamma''}(s-k)\right) \ .$$

For a proof of Theorem 14, see [2].

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, Geometry of Algebraic Curves, vol. I, 1985.

[2] D. Eisenbud, M. Green, and J. Harris, Cayley-Bacharach Theorems and Conjectures [*unpublished?*], 1995.

[3] D. Eisenbud, M. Green, and J. Harris, Higher Castelnuovo Theory, *S. M. F. Astérisque* **218** (1993), 187-202.

[4] D. Eisenbud and J. Harris, Curves in Projective Space, *Les Presses de l'Université de Montréal*, 1982.

[5] P.A. Griffiths and J. Harris, Principles of Algebraic Geometry, 1978.

[6] J. Harris, Algebraic Geometry: A First Course, 1992.

[7] J. Harris, The Genus of Space Curves, *Math. Ann.* **249** (1980), 191-204.