Math Xa Midterm 2
Study Guide
Cammie Smith's Section
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1. Know how to take derivatives, using the derivative rules (power rule, exponential rule, product rule, etc.).

   (a) **Example:** Compute derivative of $x(t) = t^2 - 3t - 4$.

   (b) **Example:** Compute derivative of $y(t) = 101^t$. (You may write the derivative in terms of $y'(0)$.)

   (c) **Example:** Compute derivative of $f(x) = 100e^{20x}$.

   (d) **Example:** Compute derivative of $g(x) = e^{5x}(x^3 - 4x^2 + \frac{1}{x+3} - 2e^{-2x})$. Don't simplify it. Hint: use the product rule.

   (e) **Example:** Compute derivative of derivative of $z(x) = 7^x$. (You may write the derivative in terms of $z'(0)$.)

   (f) **Example:** Compute derivative of $h(t) = \sqrt{t}$. Hint: rewrite as a power of t.

2. Be able to check whether or not a given function is the solution to a given differential equation.

   (a) **Example:** Is $h(t) = \sqrt{t}$ a solution of the differential equation $h' = \frac{1}{2h}$?

   (b) **Example:** Is $x(t) = t^2 - 3t - 4$ a solution of the differential equation $x' = 3t$?

   (c) **Example:** Is $w(x) = (x-2)^3$ a solution of the differential equation $w' = \frac{3w}{x-2}$?

   (d) **Example:** Is $f(x) = 100e^{20x}$ a solution of the differential equation $f' = 20f$?

   (e) **Example:** Is $m(t) = \frac{1}{t-5}$ a solution of the differential equation $m' = -m^2$?

3. Be able to draw or recognize the slope field for a given differential equation.

   (a) **Example:** Draw the slope field for the differential equation $h' = \frac{1}{2h}$.

   (b) **Example:** Draw the slope field for the differential equation $x' = 3t$.

   (c) **Example:** Draw the slope field for the differential equation $w' = \frac{3w}{x-2}$.

   (d) **Example:** Draw the slope field for the differential equation $f' = 20f$.

   (e) **Example:** Draw the slope field for the differential equation $m' = -m^2$.

4. Be able to model a situation of exponential growth or decay using a function.

   (a) **Example:** Shannon invests \$10,000 in a Certificate of Deposit that matures in 6 months, at which time the CD will be worth \$10250. Assuming that interest is compounded continuously, that rates remain the same over time, and that there are no additional fees for leaving the money in the CD indefinitely, write an equation modeling the worth of Shannon's investment, $C(t)$, in terms of time t (measured in years).

   (b) **Example:** Which is larger in the previous example, $C(0.5)$ or $\frac{C(0)+C(1)}{2}$? Explain.

   (c) **Example:** Jane adopts an "ultra-fast-growing virtual rabbit colony". She is told that her initial 5 virtual rabbits will reproduce continuously and exponentially at a rate that will give her 45 virtual rabbits after 4 days. Model the number of rabbits in the virtual colony as a function $V(t)$ in terms of the time t (measured in days). Note that the number of virtual rabbits can be measured as a (not necessarily integral) decimal number.

   (d) **Example:** Which is larger in the previous example, $V(4)$ or $\frac{V(0)+V(8)}{2}$? Explain.

   (e) **Example:** You are given two large whole pies for the holidays. Your roommate, who believes in sharing, is very fond of pie. Each day he eats exactly the same fraction of the pie remaining in the fridge. You are so busy with classes that you forget that the pies are there. On the fifth day, you remember that you have two delicious pies to eat and open the fridge. You are astounded to find that there is only half of a pie left. Assuming that your roommate eats the pie continuously, model the amount of pie remaining each day as a function $P(t)$ in terms of the time t (where t is measured in days).

   (f) **Example:** Which is larger in the previous example, $P(2)$ or $\frac{P(0)+P(4)}{2}$? Explain.

5. Be able to model a situation where a variable changes using a differential equation.

   (a) **Example:** Shannon starts a savings account, which earns 2% interest annually, with a fixed initial deposit. Assuming that interest is compounded continuously, that rates remain the same over time, and that Shannon doesn't deposit or withdraw any money into or from the account, write a *differential equation* modeling the worth of Shannon's savings account, $S(t)$, in terms of time t (measured in years).

   (b) **Example:** Write a new differential equation to model the situation where Shannon puts the same deposit into her savings account initially, but withdraws \$500 continuously over the course of each year.

   (c) **Example:** For the equation in part (5b) above, if Shannon deposits \$20,000 initially, will the account ever run out? Explain.

   (d) **Example:** If, on the other hand, she deposits \$30,000 initially, what will happen then? Explain.

   (e) **Example:** Find the equilibrium solution for the equation in part (5b).

(f) **Example:** Jane decides that her ordinary ultra-fast-growing virtual rabbit colony is kind of boring. She introduces the new rule so that her virtual colony, in addition to reproducing continuously at a rate that would triple the size of the colony in just two days (if no rabbits died), her colony has its rabbits die off at a rate of 3 per day. Model the situation using a differential equation (using $V(t)$ to denote the number of virtual rabbits after t days).

(g) **Example:** In part (5f), at least how many (whole) rabbits does she need to start with to ensure that the colony doesn't die out? Explain.

(h) **Example:** Jane's brother, Simon, decides to outdo his little sister by writing a program that produces a virtual rat colony. He uses the equation $\frac{dr}{dt} = r(r-2)^3(50-r)$ to model the change in the population of virtual rats. He hasn't yet decided how many rats with which to start the colony. Help him to decide by graphing the situation for each of the initial values below, and describing what happens to the colony.

i. $r(0) = 0$

ii. $r(0) = 1$

iii. $r(0) = 2$

iv. $r(0) = 3$

v. $r(0) = 20$

vi. $r(0) = 50$

vii. $r(0) = 100$

6. Be able to solve optimization problems.

   (a) **Example:** Jane gets tired of tending virtual rabbits and decides to buy herself some real ones. She wants to make an 8 cubic meter cylindrical pen with wire mesh sides and base and with a square wooden top to keep out the rain. The wire mesh costs \$5 per square meter, and the wooden top costs \$7 per square meter. What dimensions should she make the pen, if she wants it to cost as little as possible? Assume that the side length of the square top equals the diameter of the cylinder.

   (b) **Example:** Jane also wants to know what the maximum cost would be. Give the maximum cost or explain why there isn't one.

(c) Simon and Shannon are making large ornaments by twisting strands of gold and silver tinsel around shapes that are bent out of copper wire. They start with $2\pi + 8$ feet of copper wire and enough tinsel wrap around it, and they want to bend the wire into a square with side length at least a foot and a circle with diameter at least a foot.

i. What is the largest and the smallest that the side length s of the square can be?

ii. To maximize the sum of the area contained in the circle with the area contained in the square, what dimensions should they make each shape?

iii. To minimize the sum of the area contained in the circle with the area contained in the square, what dimensions should they make each shape?

(d) **Example:** Find the global extrema of the function $f(x) = \frac{(x-4)^2}{x}$ on the interval $(0, 7]$.

i. Hint: compute the derivative.

ii. Hint: set the derivative equal to zero and solve for the x-value of the critical point(s).

iii. Hint: Use the first or second derivative test to classify the critical point(s) that are inside the specified interval.

iv. Hint: Compare the values at the critical points to the values at the endpoint(s).

v. Hint: What is the global minimum? What is the global maximum?

vi. Do both global extrema exist? Why or why not? Your explanation should mention whether or not the Extreme Value Theorem applies in this case.