Math Xa Final Exam
Study Guide

Cammie Smith's Section

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1. Taking Derivatives

   (a) **Example:** Differentiate $f(x) = 3\pi e^6$ using any method you like. Do **not** simplify your answer.

   (b) **Example:** Differentiate $g(x) = \sqrt[5]{x^3} - \sqrt[7]{x^4}$ using any method you like. Do **not** simplify your answer.

   (c) **Example:** Differentiate $h(y) = \frac{3}{y^2} - 2\pi + e^{-4y}$ using any method you like. Do **not** simplify your answer.

   (d) **Example:** Differentiate $x(t) = e^{3t}(t^3 + 7t)$ using any method you like. Do **not** simplify your answer.

   (e) **Example:** Differentiate $z(t) = \frac{(t^4 - 6t^2)(e^{-t} + e^t)}{\sqrt{t}}$ using any method you like. Do **not** simplify your answer.

2. Composition of Functions

(a) **Example:** Consider the following table of values.

x	$f(x)$	$g(x)$	$h(x)$
-3	-2	3	3
-2	0	-3	-3
-1	2	0	2
0	3	1	-2
1	1	-2	1
2	-1	-1	-1
3	-3	0	0

i. Find $f(g(h(0)))$.

ii. Find $g(g(h(1)))$.

iii. Find $f(h(g(2)))$.

iv. Could $g(x)$ be the inverse function of $f(x)$? Explain why or why not.

v. Could $h(x)$ be the inverse function of $f(x)$? Explain why or why not.

(b) **Example:** Suppose the function $w(t)$ has domain $(-\infty, 0]$ and range $[-3, 6]$. Let $x(t) = 7 + w(3t)$ and $y(t) = 4 - w(t-2)$.

i. What is the domain of $x(t)$?

ii. What is the range of $x(t)$?

iii. What is the domain of $y(t)$?

iv. What is the range of $y(t)$?

3. Roots of Polynomials

   (a) **Example:** Suppose that $f(x)$ is a polynomial with 5 distinct roots (zeros) and that $g(x)$ is a polynomial with 7 distinct roots (zeros).

   If you cannot determine the answer for any of the following, explain why the the answer is undetermined or does not exist.

      i. What is the minimum degree of $f(x)$?

      ii. What is the maximum degree of $f(x)$?

      iii. What is the minimum degree of $g(x)$?

      iv. What is the maximum degree of $g(x)$?

      v. How many roots does $f(x+4)$ have?

      vi. How many roots does $f(x)+4$ have?

      vii. How many roots does $f(x)+g(x)$ have?

      viii. What is the minimum degree of $f(x)+g(x)$?

4. Inverse Functions

   (a) **Example:** Suppose $f(x) = \frac{4x+1}{3x-2}$.

      i. For what values of x is $f(x)$ defined? (This will be the domain of $f(x)$.)

      ii. What values does $f(x)$ take on? (This will be the range of $f(x)$.)

iii. Find a formula for the inverse function $f^{-1}(x)$.

iv. What is the domain of $f^{-1}(x)$?

v. What is the range of $f^{-1}(x)$?

(b) **Example:** Suppose $g(x) = \frac{2x}{x+3}$.

i. For what values of x is $g(x)$ defined? (This will be the domain of $g(x)$.)

ii. What values does $g(x)$ take on? (This will be the range of $g(x)$.)

iii. Find a formula for the inverse function $g^{-1}(x)$.

iv. What is the domain of $g^{-1}(x)$?

v. What is the range of $g^{-1}(x)$?

5. Sign Analysis of Derivative and Tangent Line Approximations

(a) **Example:** Suppose that a jogger begins a workout along a straight path by running at a speed of 12 miles per hour and decelerating. He decelerates for the first 10 minutes until he reaches a comfortable speed, jogs at a constant rate for the next 45 minutes, and then accelerates for the last 5 minutes of his hour-long workout, reaching a sprinting speed of 15 miles per hour just as he crosses the finish line. Let $s(t)$ denote the jogger's position (in miles from the starting point) at time t (measured in hours).

i. Find the signs of $s(0)$, $s'(0)$, and $s''(0)$, determining quantities numerically if possible.

ii. Find the signs of $s(0.25)$, $s'(0.25)$, and $s''(0.25)$, determining quantities numerically if possible.

iii. Find the signs of $s(0.96)$, $s'(0.96)$, and $s''(0.96)$, determining quantities numerically if possible.

iv. Using a tangent line approximation, estimate the distance traveled in the first 5 minutes of the workout. Will this be an over- or an under-estimate?

v. Using a tangent line approximation, estimate the distance traveled in the last 2 minutes of the workout. Will this be an over- or an under-estimate?

6. Differential Equations

   (a) **Example:** Suppose that advertisement for a new type of candy has a catchy jingle (short song to remember the product). In fact, the tune is so contagious that it spreads throughout the population of an isolated town much like a disease.

      i. Assuming that the town has a population of P inhabitants, and that in the timeframe discussed no one enters or leave the town, use a differential equation to model the following situations. Let $J(t)$ denote the number of people who remember the jingle at time t (measured in days).

         A. People hear the jingle on the radio at a constant rate and learn it immediately. Once a person learns the tune, he or she forgets it at a rate proportional to the number of people in the town who know the tune. When a person has forgotten the tune, he or she will learn it again the next time the person hears it.

         B. Once someone has heard the jingle at least one time, he or she cannot stop singing it, and so knowledge of the jingle spreads when someone who has heard it meets someone who has not.

         C. Knowledge of the jingle spreads when someone who has heard it meets someone who has not, but people forget the jingle at a rate proportional to the number of people in the town who currently know it. When a person has forgotten the tune, he or she will learn it again the next time the person hears it.

         D. People learn the tune at a rate proportional to the number of people in the town who already know it. No one is able to forget the tune in the timeframe we are modeling.

      ii. What are the units in each of the equations in part (6(a)i) above?

iii. Find a solution to the equation from part (6(a)iA) above.

iv. Compute $\lim_{t\to\infty} J(t)$, where $J(t)$ is the solution found in part (6(a)iii) above. Interpret what this computation means in light of the example (that is, in terms of the spread of the jingle).

(b) **Example:** An as yet undiscovered radioactive isotope of Carbon occurs on the as yet undiscovered Planet Xanadu. This isotope occurs in all the alien life forms of Planet Xanadu, and once an alien dies the isotope decays very rapidly, at a rate proportional to the amount of the isotope remaining in its body. A wandering interplanetary explorer happens upon a the planet and finds evidence of an extinct advanced civilization. The explorer wants to know how long ago the civilization died out. Having discovered the strange isotope of Carbon, she performs experiments on the vegetation still living on the planet, and determines the constant of proportionality is approximately -0.5.

i. Write a differential equation to describe the rate of change of $C(t)$, the amount of the strange isotope remaining in a given dead life form t years after death.

ii. A fullgrown WillyNilly plant has 500 milligrams of the strange isotope when it dies. Find a solution to the equation in part (6(b)i) above that describes the amount of isotope remaining in the plant as a function of time.

iii. Find the amount of the strange isotope remaining in the fullgrown WillyNilly plant 2 years after death.

iv. What will happen in the long term for the solution that you found in part (6(b)ii) above? Interpret this behavior in terms of the problem.

v. The explorer finds the remains of an alien that appears to be an advanced life form. She measures that there are 1,000 milligrams of the isotope remaining in the alien, and assumes that a living alien of the same form would have approximately 10,000 milligrams of the isotope. Write an equation to find out about how long ago the alien died. Do **not** solve your equation.

vi. Plot the first quadrant of the slope field for the differential equation in part (6(b)i) above.

(c) **Example:** The owner of a small convenience store, HappyMart, received a large initial stock of Twinkies for HappyMart's Grand Opening. Residents of the town adore Twinkies and consume them continuously at a rate proportional to the amount of Twinkies stocked at the local HappyMart (with a constant of proportionality equal to 0.6). The owner restocks the Twinkies in his store continuously at a rate of 3,000 Twinkies per week.

i. Write a differential equation modeling the situation, where $T(t)$ denotes the number of Twinkles stocked at the store t weeks after the Grand Opening.

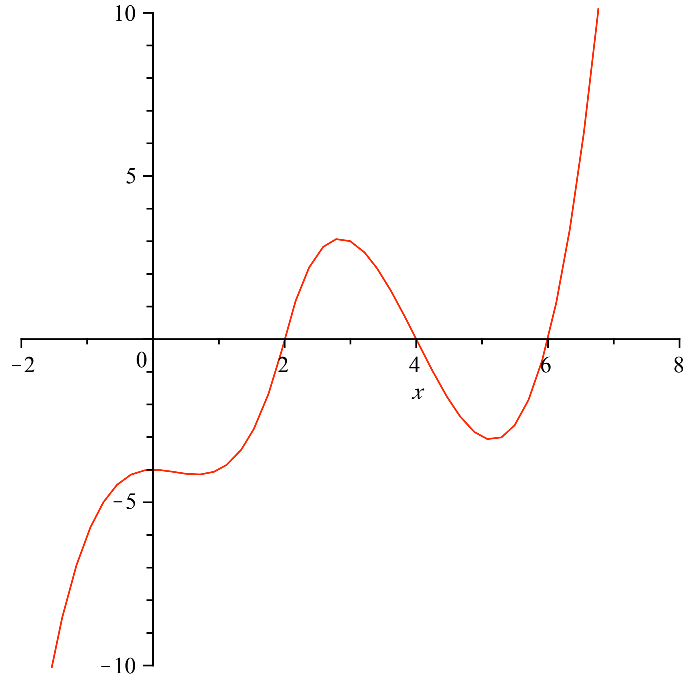
ii. Suppose that the number of Twinkies in the store remains constant with time. How large was the initial shipment of Twinkies?

iii. If the initial shipment instead contains 4,000 Twinkies, will HappyMart ever run out?

iv. What happens if the initial shipment contains 6,000 Twinkies?

v. On a single coordinate plane, graph the situation in each of parts (6(c)ii), (6(c)iii), and (6(c)iv) above. Label any equilibrium solutions.

(d) **Example:** Suppose that $\frac{dx}{dt} = h(x)$, where $h(x)$ is the function whose graph is below.



On the coordinate system below, sketch a graph for each of the following initial conditions. Label any equilibrium solutions

i. $x(0) = 1$

ii. $x(0) = 2$

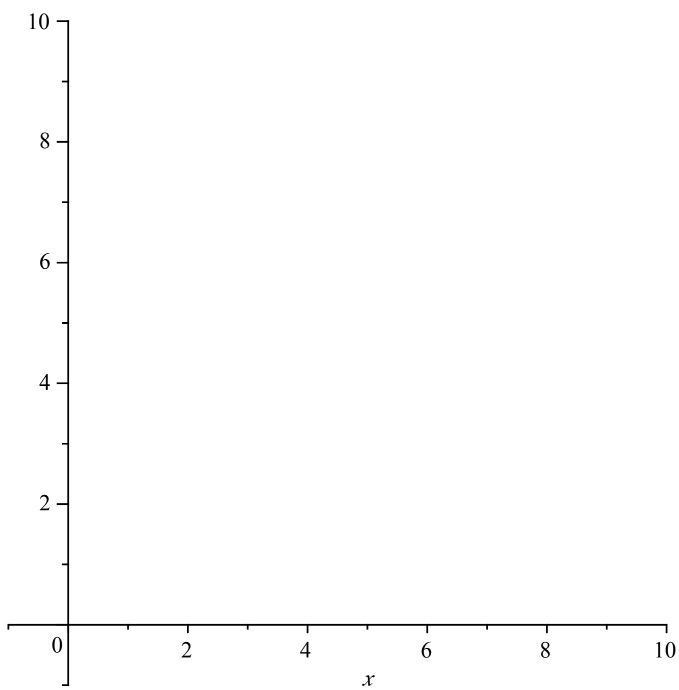
iii. $x(0) = 3$

iv. $x(0) = 4$

v. $x(0) = 5$

vi. $x(0) = 6$

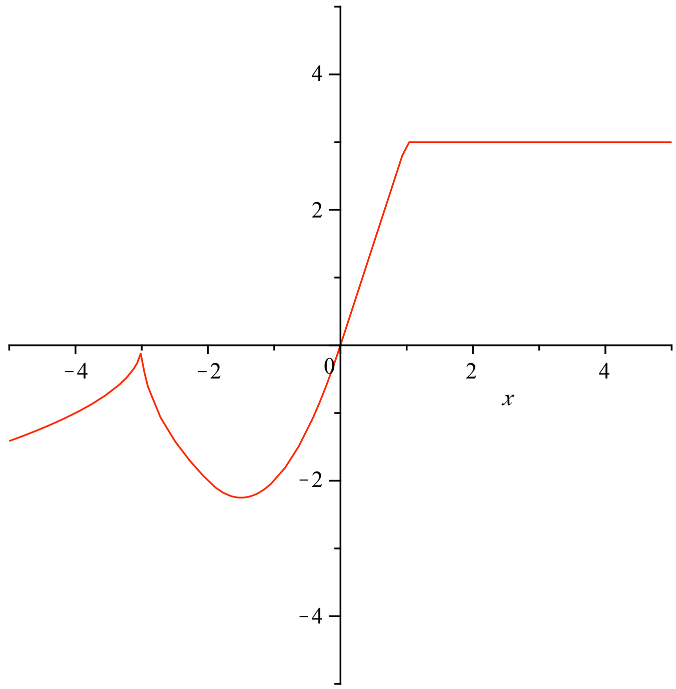
vii. $x(0) = 7$



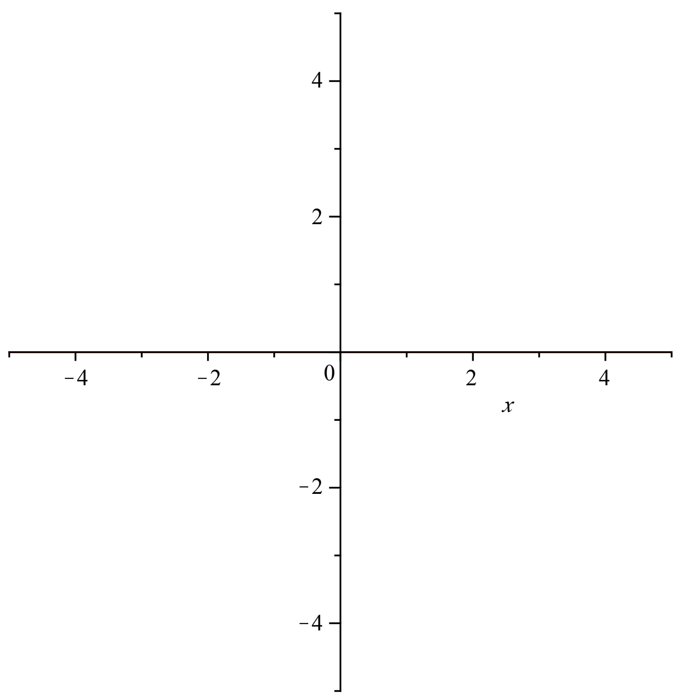
7. Graphing Functions

(a) **Example:** Using the Derivative in Graphing

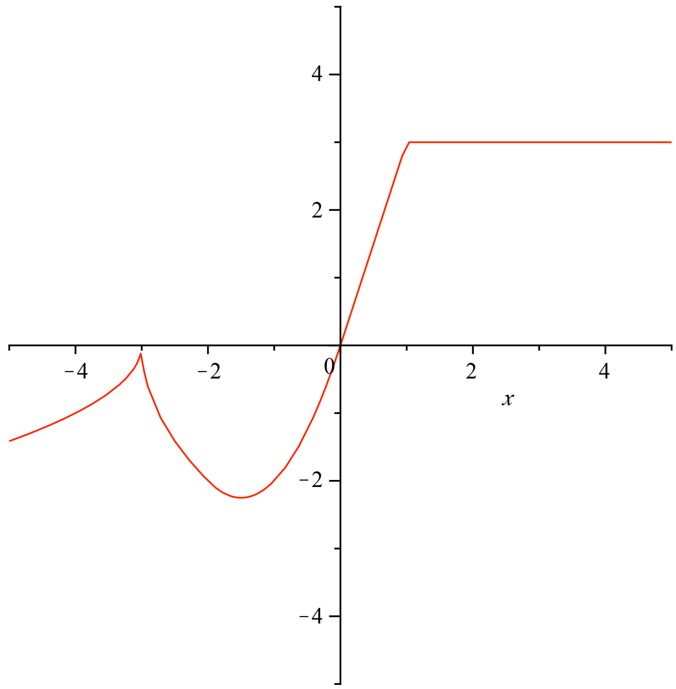
i. Suppose that the following is the graph of $f(x)$.



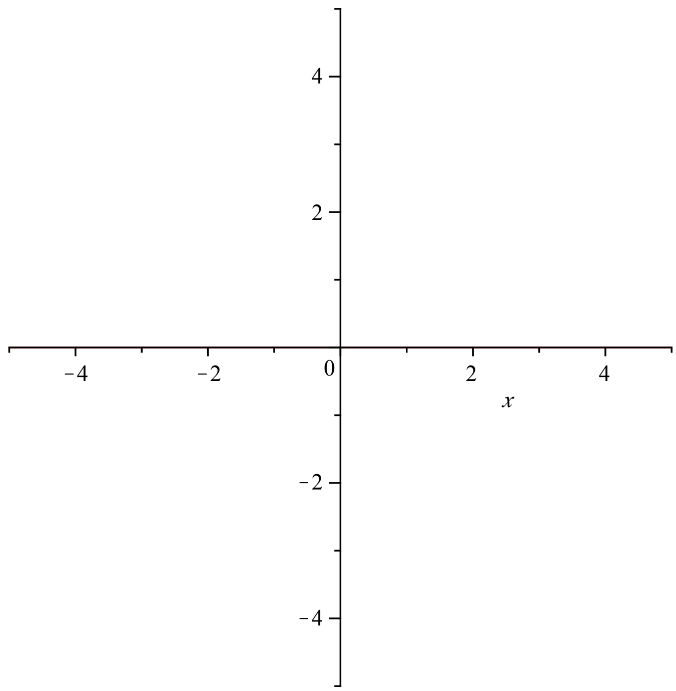
Sketch the graph of $f'(x)$.



ii. Suppose that the following is the graph of $g'(x)$.



Sketch the graph of $g(x)$, where $g(0) = -4$.



(b) **Example:** Rational Functions

Suppose that

$$h(x) = \frac{(x+2)(x-3)}{x(x-1)}.$$

Then

$$h'(x) = \frac{6(2x-1)}{x^2(x-1)^2}.$$

(Do **not** verify this.)

i. At which values of x are the roots of $h(x)$?

ii. At which values of x are the roots of $h'(x)$?

iii. Compute $\lim_{x\to\infty} h(x)$.

iv. Compute $\lim_{x\to\infty} h'(x)$.

v. Compute $\lim_{x\to-\infty} h(x)$.

vi. Compute $\lim_{x\to-\infty} h'(x)$.

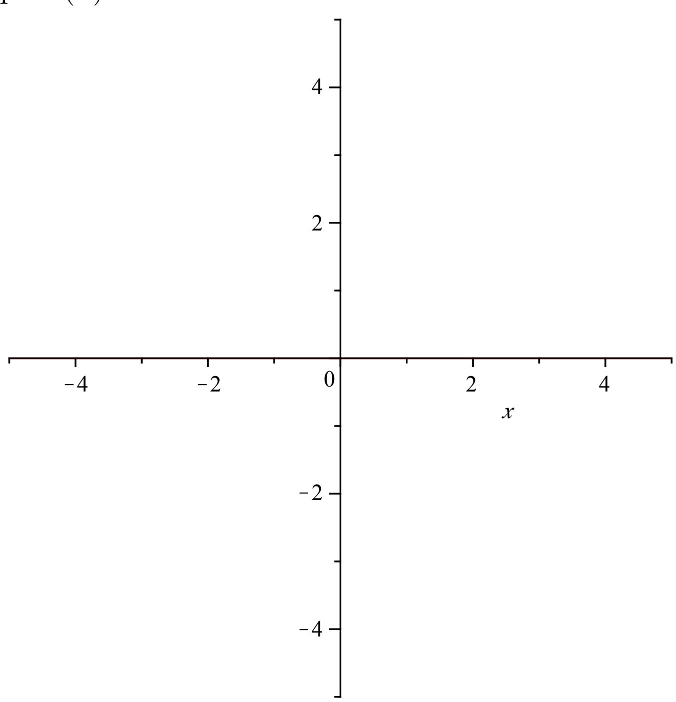
vii. At which values of x does $h(x)$ have a vertical asymptote?

viii. At which values of x does $h'(x)$ have a vertical asymptote?

ix. At which values of x does $h(x)$ change sign?

x. At which values of x does $h'(x)$ change sign?

xi. Graph $h(x)$ on the axes below.



8. Optimization

Example: Jillian is building a model farm out of clay. For sturdiness and ease of sculpting, she makes all her buildings out of solid clay. She has $c = \frac{243\pi^4}{4}$ cubic centimeters of clay remaining, but still needs to build a barn and a silo.

She wants the barn to be a rectangular prism surmounted by a triangular prism of equal width, so that the front face of the barn is a rectangle surmounted by an isosceles triangle of height and base equal to the height and base, respectively, of the rectangle, with the base of the rectangle being twice its height. She has decided that the height of the rectangle will be $h = 2\pi + \frac{4\pi\sqrt{2}}{3}$ centimeters.

Her silo needs to be a cylinder surmounted by a hemisphere of radius equal to the radius of the cylinder. She has decided that the height of the cylinder will be $k = 6\pi$ centimeters.

(a) Write a constraint that the length of the barn and the radius of the silo must satisfy. *Hint: The constraint should involve the total volume of the clay.*

(b) When the clay in her model dries, she will paint the hemisphere of the silo and the vertical faces of the barn red, using white paint for the remaining surfaces. Suppose that red paint costs 1.5 times as much as white. How can she build the model so as to minimize the cost of the paint used?