Irreducible algebraic sets

Let $X$ be an algebraic set.

**Def:** $X$ is **reducible** if $X = X_1 \cup X_2$ where $X_1, X_2 \not\subseteq X$ are algebraic sets. Otherwise $X$ is irreducible.

**Ex:** $V(xy) = V(x) \cup V(y)$, but the only algebraic sets in $V(x)$ are finite, so $V(x)$ is irreducible.

**Def:** If $X = X_1 \cup \ldots \cup X_m$, where each $X_i$ is irreducible and $X_i \not\subseteq X_j$ for $i \neq j$, the $X_i$'s are called **irreducible components** of $X$.

We can always find such a decomposition:

**Thm:** $X$ an algebraic set.

a.) We can write $X = X_1 \cup \ldots \cup X_m$ where the $X_i$ are irreducible components.

b.) The decomposition in a.) is unique.

**Pf:** a.) If $X$ is irreducible, we're done. Otherwise $X = Y \cup Z$, $Y$ and $Z$ both proper algebraic subsets.

We can continue by decomposing $Y$ or $Z$, stopping when...
all alg. sets are irreducible.

If the process never stops, we get an infinite sequence

\[ X \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots \]

which can't happen by the Hilbert Basis Theorem.

6.) Suppose \( X = X_1 \cup \ldots \cup X_r \) and \( X = Y_1 \cup \ldots \cup Y_s \) are two irreducible decompositions.

For each \( X_i \), we can write \( X_i = \bigcup_{j=1}^{s} (Y_j \cap X_i) \).

Since \( X_i \) is irreducible, \( X_i \subseteq Y_j \), some \( j \).

Similarly, \( Y_j \subseteq X_k \), some \( k \). \( \Rightarrow \) \( X_i \subseteq X_k \), so \( i = k \)

\( \Rightarrow \) \( X_i = Y_j \). \( \square \)

We already know that each algebraic set \( X \) gives us an ideal \( I(X) \). If \( X \) is irreducible, we can say more:

**Prop:** \( X \) is irreducible \( \iff \) \( I(X) \) is prime.

(Recall \( J \) is prime if for \( f, g \in J \), \( fg \in J \) or \( g \in J \).

**Pf:** Assume \( X \) is reducible. Then \( X = X_1 \cup X_2 \), and
\[ I(X_1), I(X_2) \nsubseteq I(X). \]

Since \( x_1 \not\in X \), let \( f_i \in I(X_i) \setminus I(X) \).

Then for \( P \in X \), \( f_i(P) = 0 \) or \( f_2(P) = 0 \) \( \Rightarrow (f_1f_2)(P) = 0 \).

\[ \Rightarrow f_1f_2 \in I(X) \text{ so } I(X) \text{ is not prime.} \]

Now assume \( I(X) \) is not prime. Then \( \exists f, g \notin I(X) \) s.t. \( fg \in I(X) \).

\[ \Rightarrow X \subseteq V(fg) = V(f) \cup V(g) , \text{ but } X \notin V(f) \text{ or } V(g). \]

\[ \Rightarrow X = (V(f) \cap X) \cup (V(g) \cap X) \Rightarrow X \text{ is reducible.} \]

Q: If \( J \) is a prime ideal, is \( V(J) \) irreducible?

No:

Ex: Consider \( f = y^2 + x^2(\alpha - 1)^2 \in \mathbb{R}[x,y] \)

\( f \) is irreducible, so \( (f) \) is prime (exer)

But \( V(f) = \{(0,0), (1,1)\} = V(x, y) \cup V(\alpha - 1, y). \)

Dimension: \( k = \text{alg. closed} \)

If \( X \subseteq \mathbb{A}^n \) is an alg. set, we can write \( X \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_0 \supseteq \emptyset \)

where \( X_i \) is an irr. alg. set.
The maximum such $d$ is the **dimension** of $X$.

**Ex:** $\dim A^f = 1$. We know $\dim A^n \geq n$, but we can't quite show equality yet.

More generally (over any $k$), $\dim X =$ maximum length of chain of prime ideals containing $I(X)$. These agree when $k = \overline{k}$.