We'll hear a lot more about specifics of divisors on curves in the final presentations, but here is a rough intro to divisors and linear equivalence.

**Motivation:** In $\mathbb{P}^n$, we can call an irreducible hypersurface $S$ of deg $d$ a “prime divisor.” Then $|S|$ is the set of divisors “linearly equivalent” to $S$. It turns out $|S| = |dH|$, i.e. the complete linear systems are in bijection w/ the possible degrees.

We want to generalize the notion of linear system to arbitrary projective varieties, so we need to define what we mean by a divisor and linear equivalence of divisors.

(Also called Weil divisors in order to distinguish from Cartier divisors. On a smooth variety, these notions agree.)

**Def:** Let $X$ be a smooth variety. A divisor on $X$ is a formal sum $$D = a_1 D_1 + a_2 D_2 + \cdots + a_r D_r$$

where $a_i \in \mathbb{Z}$, and each $D_i$ is a closed irreducible subvariety of $X$ of codimension 1.

$D$ is effective if $a_i \geq 0 \ \forall \ i$.

$a_1 = 1$, and the rest of the $a_i = 0$, $D$ is called a prime divisor.
The divisors on $X$ naturally form an additive group, the free abelian group generated by the codim 1 subvarieties, called $\text{Div} X$.

This group isn't very interesting by itself, but it has a very interesting quotient.

**Divisors from rational functions**

Let $X$ be a projective (or affine) variety.

Let $f = \frac{g}{h} \in k(X)$. We want to define a divisor $\text{div}(f)$ s.t., roughly, $\text{div}(f) = \sum a_i C_i - \sum b_i D_i$, where $C_i$ are the components of $g$, and $a_i$ is the "order of vanishing" of $g$ along $C_i$, and similarly for $b_i, D_i, h$.

We need to figure out how to assign coefficients.

Let $C$ be a prime divisor on $X$. We define the integer $v_C(f)$ as follows:

1. **Assume** $\Gamma(C_\times)$ is a UFD for this. In general we need to choose a different open cover, but this is the rough strategy:
2. Choose a $U_i$ s.t. $C \notin X \setminus U_i$. If $X$ is affine, skip this step. For smooth $X$, $C_\times = V(\alpha)$, some $\alpha \in \Gamma(X_\times)$.
3. Define the local ring $O_{C_\times}(X_\times) = \{ \frac{a}{b} \in k(X_\times) \mid b \not\equiv (\alpha_\times) \}$
   
   This is a DVR with maximal ideal $(\alpha_\times)$. 


\( v_c(f) = \text{valuation of image of } f \text{ in } k(X) \text{ w.r.t. uniformizing parameter } \alpha. \) (claim: \( v_c(f) \) doesn't depend on choice of \( U_i \).)

Roughly: If \( f = \frac{g}{h} \),
\[ v_c(f) = (\text{order of vanishing of } g \text{ along } C) - (\text{order of vanishing of } h \text{ along } C) \]

**Def:** \( \text{div}(f) = \sum v_c(f) C, \ C \subseteq X \text{ a prime divisor.} \)

Since in any \( U_i, g_x \) has only finitely many irreducible components, \( v_c(f) \neq 0 \) for only finitely many \( C \).
\[ \Rightarrow \text{div}(f) \in \text{Div}(X). \]

**Def:** \( P(X) = \{ \text{div}(f) \mid f \in k(X) \} \subseteq \text{Div}(X) \) is the set of principal divisors.

\( P(X) \) is a subgroup of \( \text{Div}(X) \), and \( \text{Cl}(X) = \text{Div}(X)/P(X) \)

is the divisor class group of \( X. \)

A coset of \( \text{Cl}(X) \) is called a divisor class. If \( D_1 \) and \( D_2 \) are in the same divisor class, they are linearly equivalent, written \( D_1 \sim D_2. \)

**Ex:** What is \( \text{Cl}(\mathbb{A}^n) \)? Let \( V \subseteq \mathbb{A}^n \) be an irreducible hypersurface, then \( V = V(f), \) some \( f \in \Gamma(\mathbb{A}^n) = k[x_1, \ldots, x_n]. \)
But \( k(A^n) \) is \( k(x_1, \ldots, x_n) \), so \( V = \text{div} \left( \frac{f}{h} \right) \).

Thus, \( V \in P(A^n) \), for all prime divisors.

So \( P(A^n) = \text{Div}(A^n) \Rightarrow \text{Cl}(A^n) = 0 \).

**Ex:** \( \text{Cl}(P^n) \)?

Define the degree of a divisor \( \sum a_i D_i \in \text{Div}(P^n) \) to be \( \sum a_i \deg(D_i) \).

If \( f \in k(P^n) \), then \( f = \frac{g}{h} = \frac{g_i}{h_i^{b_i}} \cdots \frac{g_m}{h_i^{b_m}} \) where \( \deg g = \deg h \) and \( g_i, h_j \) irreducible.

Then \( \text{div}(f) = \sum a_i V(g_i) - \sum b_j V(h_j) \Rightarrow \deg \left( \text{div}(f) \right) = 0 \)

Moreover if \( D = \sum a_i V_i \) has deg 0 and \( V_i = V(f_i) \), \( f_i \) homog, then \( f = \prod f_i^{a_i} \in k(V) \Rightarrow D = \text{div}(f) \).

Thus, \( \text{Cl}(X) = \mathbb{Z} \).

**Linear Systems**

Let \( X \) be a smooth variety, and \( D \in \text{Div}(X) \).

**Def:** The linear system of \( D \), denoted \(| D |\) is
\(|D| = \{D' \in \text{Div}(X) \mid D' \text{ is effective and } D \sim D'\}\)

In other words, this is the projectivization of the vector space
\[ \mathcal{L}(D) = \{ f \in k(X) \mid \text{div } f + D \geq 0 \} \cup \{0\} \]
i.e. if \( D = \sum n_i C_i \) then \( f \in \mathcal{L}(D) \) if
\[ \nu_{C_i}(f) \geq -n_i \text{ and } \nu_{C}(f) \geq 0 \text{ for } C \neq C_i. \]

**Exercise**: The properties of valuations tell us that this is a vector space.

**Example**: If \( H \) is any hyperplane in \( \mathbb{P}^n \),
\[ |H| = \{ H' \mid H' \text{ is a hyperplane} \}, \]
and
\[ |dH| = \begin{cases} \{ \text{hyperpts of deg } d \} & \text{for } d > 0 \\ \emptyset & \text{for } d < 0 \end{cases} \]

A complete linear system on \( X \) is \(|D|\) for some divisor \( D \).

If \( V \subseteq \mathcal{L}(D) \) is a linear subspace, \(|V| \subseteq |D|\) is the corresponding linear system.