**Projective algebraic sets**

A point $P \in \mathbb{P}^n$ is a **zero** of a polynomial $f \in k[x_1, \ldots, x_{n+1}]$ if $f(x_1, \ldots, x_{n+1}) = 0$ for every choice $P = [x_1 : \cdots : x_{n+1}]$ of homogeneous coordinates for $P$.

**Example:** $f(x, y) = x - y + 1$. Then if $P = [2:1] = [4:2]$ then $f(2, 1) = 0$, but $f(4, 2) \neq 0$, so $P$ is not a zero of $f$.

**Claim:** If $f$ is a form and $f$ vanishes at one representative of $P$, then $f$ vanishes at all representatives of $P$.

**Proof:** $f$ a form of deg $d$. Let $P = [x_1 : \cdots : x_{n+1}] = [\lambda x_1 : \cdots : \lambda x_{n+1}]$.

Suppose $f(x_1, \ldots, x_{n+1}) = 0$.

Then $f(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d f(x_1, \ldots, x_{n+1}) = 0$. □

Let $f \in k[x_1, \ldots, x_{n+1}]$. Write $f = f_0 + \cdots + f_d$ where $f_i$ is homogeneous of degree $i$. Then $P \in \mathbb{P}^n$ is a zero of $f \iff P$ is a zero of $f_i \ \forall \ i$. (on HW)

**Def:** Let $S \subseteq k[x_1, \ldots, x_{n+1}]$. \(V(S) = \{ P \in \mathbb{P}^n \mid P \text{ is a zero of each } f \in S \}\)

$V(S)$ is called a **projective algebraic set**.
Note:

1. If $I$ is the ideal generated by $S$, then $V(I) = V(S)$.

2. If $I = (f_1, \ldots, f_r)$ and $f_i = \sum f_{ij}$, then $V(I) = V(\{f_{ij}\}_{ij})$

   form of deg $j$

So $V(S)$ is the set of zeros of a finite # of forms.

**Def:** Let $X \subseteq \mathbb{P}^n$. The ideal of $X$ is

$$I(X) = \{ f \in k[x_1, \ldots, x_{n+1}] \mid \text{every } P \in X \text{ is a zero of } f \}$$

**Note:** $I(X)$ is homogeneous.

**Ex:** $(x + y^2)$ is not homogeneous.

**Prop:** An ideal $I \subseteq k[x_1, \ldots, x_{n+1}]$ is homogeneous $\iff$ it's generated by a (finite) set of forms.

**Pf:** Let $I = (f_1, \ldots, f_r)$ and $f_i = \sum f_{ij}$ and suppose $I$ is homogeneous.

Then $f_{ij} \in I \forall i,j$ and $I \subseteq (f_{ij})_{ij} \Rightarrow I = (f_{ij})_{ij}$. 
Now suppose \( I = (f_1, \ldots, f_r) \), each \( f_i \) a form of deg \( d_i \).

Suppose \( g = g_m + g_{m+1} + \ldots + g_s \in I \), \( g_j \) a form of deg \( j \).

We show \( g_j \in I \) by induction on \( m \), where the base case is \( m = s \), i.e.
\( g \) is already homogeneous.

i.e. assume true when smallest deg of \( g \geq m \).

\[ g = \sum a_i \cdot f_i. \] Since each \( f_i \) is a form, we can write

\[ a_i \cdot f_i = (a_{i0} + a_{i1} + \ldots) \cdot f_i = a_{i0} \cdot f_i + a_{i1} \cdot f_i + \ldots \]

so \( g_m = a_{1,m-d_1} \cdot f_1 + \ldots + a_{r,m-d_r} \cdot f_r \in I \).
Thus, \( g - g_m \in I \). Done by induction. \( \square \)

**Remark:** Any projective set can be written \( V(I) \), \( I \) homogeneous, and \( I(x) \) is homogeneous, so we have

\[ \left\{ \text{homogeneous ideals in } \mathbb{K}[x_1, \ldots, x_m] \right\} \xrightarrow{V} \left\{ \text{projective algebraic sets in } \mathbb{P}^n \right\} \]

(not necessarily one-to-one) satisfying analogous conditions as in the affine case.

**Ex:** Points in \( \mathbb{P}^2 \): Let \( P = [a:b:c] \in \mathbb{P}^2 \).
WLOG, \( c = 1 \).
P = [a:b:1]. Let I = (a \z - x, b \z - y).
P \in V(I) and if Q = [x:y:z] \in V(I), then Q = [a \z:b \z:z] 
So \z \neq 0 \implies Q = P.

**Def:** An algebraic set \( V \subseteq \mathbb{P}^n \) is irreducible if it's not the union of two smaller algebraic sets. An irreducible alg. set in \( \mathbb{P}^n \) is a projective variety.

**Claim:** \( V \subseteq \mathbb{P}^n \) irreducible \iff I(V) is prime.

**Pf:** Essentially the same as in the affine case.

**Affine cones**
Let \( V \subseteq \mathbb{P}^n \) be an algebraic set.

**Def:** The affine cone over \( V \) is

\[
C(V) = \{(x_1, ..., x_{n+1}) \in \mathbb{A}^{n+1} \mid [x_1: ...: x_{n+1}] \in V\} \cup \{0\}
\]

i.e. the union of the corresponding lines in affine space.

**Ex:** 1) \( V = \{[1:0], [1:1], [0,1]\} \subseteq \mathbb{P}^1 \)

\[
C(V) = l_1 \cup l_2 \cup l_3
\]
2.) $I = (x^2 + y^2 - z^2) \subseteq \mathbb{C}[x, y, z], \quad V = V_p(I)$

\[ \uparrow \text{to denote projective} \]

In $\mathbb{A}^3$, this is $x^2 + y^2 = 1$.

In $\mathbb{H}^3$, this is $(x+iy)(x-iy) = 0$, i.e. 2 points

**Remark:** 1.) If $V \neq \emptyset$, then $I_a(C(V)) = I_p(V)$

2.) If $I$ is a homogeneous ideal in $k[x_1, \ldots, x_{n+1}]$ s.t. $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$

**Thm:** (Projective Nullstellensatz) Let $I$ be a homogeneous ideal in $k[x_1, \ldots, x_{n+1}]$. Then

1.) $V_p(I) = \emptyset \iff \exists$ an integer $N$ s.t. $I$ contains all forms of deg $\geq N$.

2.) If $V_p(I) \neq \emptyset$, then $I_p(V_p(I)) = \sqrt{I}$.

**Pf:** 1.) First we reduce to a question about affine varieties.

If $V_p(I) \neq \emptyset$, then $V_a(I) = C(V_p(I)) \supseteq \{(0, 0, \ldots, 0)\}$.

If $V_p(I) = \emptyset \Rightarrow V_a(I) \setminus \{(0, \ldots, 0)\} = \emptyset \Rightarrow V_a(I) \subseteq \{(0, \ldots, 0)\}$

So we want to show $V_a(I) \subseteq \{(0, \ldots, 0)\} \iff (x_1, \ldots, x_{n+1})^N \in I$, some $N \geq 1$.

If $(x_1, \ldots, x_{n+1})^N \in I$ then $V_a(I) \subseteq V((x_1, \ldots, x_{n+1})^N) = \{(0, \ldots, 0)\}$. 
If \( V_a(I) \subseteq \{(0, \ldots, 0)\} \Rightarrow (x_1, \ldots, x_{n+1}) \subseteq \sqrt{I} \Rightarrow \exists \nu > 0 \text{ s.t. } x_i^\nu \in I \forall i.

Let \( N = r(n+1) \). Then any monomial of deg \( N \) will be divisible by \( x_i^\nu \) for some \( i \Rightarrow (x_1, \ldots, x_{n+1})^N \subseteq I. \)

2. If \( V_p(I) \neq \emptyset \), then \( I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I}. \)

The usual corollaries hold, except we need to be careful of the irrelevant ideal \( I_{irr} = (x_1, \ldots, x_{n+1}). \)

**Cor:** Let \( S = k[x_1, \ldots, x_{n+1}] \). We have the following bijective correspondences (exer)

\[
\begin{align*}
\{ \text{algebraic sets in } \mathbb{P}^n \} & \longleftrightarrow \{ \text{homogeneous radical ideals in } S, \text{ other than } I_{irr} \} \\
\{ \text{irreducible alg. sets in } \mathbb{P}^n \} & \longleftrightarrow \{ \text{homogeneous prime ideals, other than } I_{irr} \} \\
\{ \text{irreducible hypersurfaces in } \mathbb{P}^n \} & \longleftrightarrow \{ \text{irreducible nonconstant forms, up to scaling} \}
\end{align*}
\]

The hyperplanes \( V(x_i), i = 1, \ldots, n+1, \) are the coordinate hyperplanes or the hyperplanes at infinity w.r.t. \( U_i. \)

**Ex:** In \( \mathbb{P}^2 \), the \( V(x_i) \) are the three coordinate axes.
Each pair intersects in one point.

Def: \( V \subseteq \mathbb{P}^n \) a projective algebraic set is \textbf{Zariski closed}. 
\( \mathbb{P}^n \setminus V \) is \textbf{Zariski open}. This gives the \textbf{Zariski topology} on \( \mathbb{P}^n \).