Intersection Numbers

let $f, g$ be plane curves and $P \in \mathbb{A}^2$. We want to define an intersection $#_I$ of $f$ and $g$ at $P$, $I(P, f \cap g)$.

What would we like it to look like?

First we'll list 7 properties we want $I(P, f \cap g)$ to have. Then we'll show the properties define a unique such $#_I$.

Properties of $I(P, f \cap g)$

1) If $f$ and $g$ have a common component that passes through $P$, then $I(P, f \cap g) = \infty$. Otherwise $I(P, f \cap g) \in \mathbb{N}_{\geq 0}$

2) a) $I(P, f \cap g) = 0 \iff P \notin f \cap g.$

   b) $I(P, f \cap g)$ depends only on the components of $f$ and $g$
that pass through $P$.

$I(P, f \cap g) = 0$

$I(P, f \cap g) = I(P, f_i \cap g_i)$

3.) If $T$ is an affine change of coordinates and $T(Q) = P$, then

$I(P, f \cap g) = I(Q, T^*f \cap T^*g)$

4.) $I(P, f \cap g) = I(P, g \cap f)$

5.) $I(P, f \cap g) \geq m_p(f) m_p(g)$, with equality occurring if and only if $f$ and $g$ have no tangent lines in common at $P$.

$I(P, f \cap g) = 1$

$I(P, f \cap g) > 1$

$I(P, f \cap g) = 2 \cdot 2 = 4$

**Def:** $f$ and $g$ intersect transversally at $P$ if $P$ is a simple point on both $f$ and $g$ and the tangent line to $f$ at $P \neq$ tangent line to $g$ at $P$. (i.e. $f$ and $g$ intersect transversally at $P \iff I(P, f \cap g) = 1$)

6.) If $f = \prod f_i^{r_i}, g = \prod g_j^{s_j}$, then

$I(P, f \cap g) = \sum_{i,j} r_i s_j I(P, f_i \cap g_j)$

7.) $I(P, f \cap g) = I(P, f \cap (g + af))$ for $a \in k[x, y]$, i.e. the intersection number depends only on the image of $g$ in $\Gamma(f)$. 
**Lemma:** Any intersection number satisfying these properties is unique.

**Pf:** Assume \( I(P, f \land g) \) is a \# satisfying 1.) - 7.). We give a procedure for calculating it:

By 3.), we can assume \( P = (0,0) \), and by 1.), we can assume it's finite.

By 2.), we know when \( I(P, f \land g) = 0 \).

Assume \( I(P, f \land g) = n > 0 \) and \( I(P, a \land b) \) can be calculated using these properties for \( I(P, a \land b) < n \).

Consider \( f(x,0), g(x,0) \in k[x] \) of deg \( r, s \) respectively. (Say \( r \) or \( s = 0 \) if the corr. poly. is zero.)

By 4.), we can assume \( r \leq s \).

**Case 1:** \( r = 0 \). Then \( y \) divides \( f \), so \( f = yh \)

\[
\Rightarrow I(P, f \land g) = I(P, y \land g) + I(P, h \land g) \\
\forall \ (0) \Rightarrow \bigwedge_{n} \text{Sinu Pe V(g)} \quad \text{which can be calculated by induction.}
\]

What is the first summand?
Write \( g(x,0) = x^m(a_0 + a_1 x + \ldots) \), \( a_0 \neq 0 \).

\[ \Rightarrow g = A x^m + B_y, \quad A(p) \neq 0. \]

\[ \Rightarrow \text{I}(P, g \cap y) = \text{I}(P, A x^m \cap y) = \text{I}(P, A \cap y) + m \text{I}(P, x \cap y) = m. \]

Case 2: \( r > 0 \). WLOG \( f(x,0) \) and \( g(x,0) \) are monic.

Set \( h = g - x^{s-r} f \). Then \( h(x,0) = g(x,0) - x^{s-r} f(x,0) \)

so \( \deg(h(x,0)) < s \) and \( \text{I}(P, f \cap g) = \text{I}(P, f \cap h) \).

Repeating this process finitely many times (switching the orders of the curves when necessary), we end up in case 1. \( \square \)

**Ex:** Assuming such an intersection \( \# \) exists, what is \( \text{I}(P, f \cap g) \), where \( f = y - x^2 - y^2 \) and \( g = y - x^3 \), and \( P = (0,0) \)?

Let \( h = g - x f = y - x^3 - xy + x^3 + xy^2 = y - xy + xy^2 = y(1 - x + xy) \)

So \( \text{I}(P, f \cap g) = \text{I}(P, f \cap h) = \text{I}(P, f \cap (1-x+xy)) + \text{I}(P, f \cap y) \)

\[ \Rightarrow = \text{I}(P, (y - x^2 - y^2) \cap y) = \text{I}(P, x^2 \cap y) = 2. \]

**Thm:** There is a unique intersection \( \# \) satisfying \( 1.) - 7.) \), given by

\[ \text{I}(P, f \cap g) = \dim_k (\mathcal{O}_P(A^2)/(f, g)) \]
Pf: We already showed uniqueness, so we just need to show it satisfies the properties.

2.) a) \( \mathbb{P} \not\subset f \cap g \iff \exists h \in k[x, y] \setminus I \mathbb{P} (h) \neq 0. \)
\[ \iff \exists \text{ a unit in } (f, g) \subseteq \mathcal{O}_p (\mathbb{A}^2). \]
\[ \iff \dim (\mathcal{O}_p (\mathbb{A}^2)) = 0. \]

b) If \( f = f_1, f_2 \) and \( f_2 \) doesn't pass through \( \mathbb{P}_1 \), then \( f_2 \) is a unit in \( \mathcal{O}_p (\mathbb{A}^2) \)
so \( (f, g) = (f, g) \subseteq \mathcal{O}_p (\mathbb{A}^2) \)

4.) Obvious.

7.) \( (f, g) = (f, g + af) \).

3.) An affine change of coordinates induces an isomorphism of local rings, so this holds.

Thus, we may from now on assume \( \mathbb{P} = (0, 0) \) and that all components pass through \( \mathbb{P} \).

1.) Claim (Prop 6 of 2.9) if \( I \subseteq k[x, y] \) is an ideal and
\[ V(I) = \{ P_1, ..., P_n \} \text{ finite, then } k[x, y] \times I \cong \bigoplus_{i=1}^{n} \mathcal{O}_{P_i} \times I \mathcal{O}_{P_i}. \]

Recall: If \( f \) and \( g \) have no common components, they must intersect in finitely many points.
Thus, setting $\mathcal{O} = \mathcal{O}_p(A^2)$, we have $\dim_k(\mathcal{O}/(f, g)) \leq \dim_k(k[x, y]/(f, g))$ finite by a corollary of Nullstellensatz.

If $f$ and $g$ have a common component $h$ then $(f, g) \subseteq h$

\[ \Rightarrow \quad \mathcal{O}/(f, g) \to \mathcal{O}/(h) \]
\[ \dim_k(\mathcal{O}/(f, g)) \geq \dim_k(\mathcal{O}/(h)) \]

WTS $\dim_k(\mathcal{O}/(h)) = \infty$.

Consider $k[x, y] \to \Gamma(h)$.

We can extend this to the quotient $\mathcal{O}_p(A^2) \to \mathcal{O}_p(h)$.

This map is surjective, and $\frac{f}{g} \mapsto 0 \iff \frac{f}{g} = 0$ in $\mathcal{O}_p(h)$

$\iff \overline{f} = 0 \iff f \in (h)$, so the map has kernel $(\frac{h}{1})$.

Thus $\mathcal{O}_p(A^2)/(h) \cong \mathcal{O}_p(h)$.

But $\Gamma(h) \subseteq \mathcal{O}_p(h)$ is infinite dimensional $\Rightarrow \dim_k(\mathcal{O}_p(A^2)/(f, g)) = \infty$.

For 5.) and 6.), see Fulton. $\blacksquare$

**Ex:** $f = (x^2 + y^2)^3 - 4x^2y^2$ and $g = (x^2 + y^2)^2 + 3x^2y - y^3, \ P = (0, 0)$.
What is $I(P, f \cap g)$?

Notice:  
\[
f - (x^2 + y^2)g = -4x^2y^2 - (3x^2y - y^3)(x^2 + y^2)
= y\left(-4x^2y - (3x^2 - y^3)(x^2 + y^2)\right)
\]

So $I(P, f \cap g) = I(P, g \cap h) + I(P, g \cap y)$

Now we continue by using the method from the proof:

$h$ has highest deg $x$-term $-3x^4$, and $g$ has $x^9$.

So we replace $h$ by $h + 3g = -4x^2y - 3x^4 - 2x^2y + y^3 + 3(x^7 + 2x^4y + y^9 + 3x^2y - y^3)$
\[
= 5x^2y + 4x^2y + y^3 - 3y^3
= y\left(5x^2 + 4x^2 + y^3 - 3y^3\right)
\]

$\Rightarrow I(P, f \cap g) = I(P, g \cap q) + 2I(P, g \cap y)$

tangent lines of $q$ are $\sqrt{5}x \pm \sqrt{3}y$, so $q$ and $g$ have no tangent lines in common, so

$I(P, g \cap q) = m_p(g) \cdot m_p(q) = 3 \cdot 2 = 6$.

$I(P, g \cap y) = I(P, x^4 \cap y) = 4$, so $I(P, f \cap g) = 6 + 2 \cdot 4 = 14$.

One more property of the intersection number:

*Prop*: If $P$ is a simple point of $f$, then $I(P, f \cap g) = \text{ord}_p(f)$
We can assume $f$ is irreducible (by forgetting other components).

Since $\mathcal{O}_p(f)$ is a DVR, $\text{ord}_p(g) = \dim_k (\mathcal{O}_p(f)/(g))$.

$$\mathcal{O}_p(A^1)/(f) \cong \mathcal{O}_p(f), \text{ so } \mathcal{O}_p(A^2)/(f,g) \cong (f)/(g).$$