You may assume \( k \) is algebraically closed.

1. Let \( F \) be an irreducible projective plane curve. Show that \( F \) has only finitely many singular points.

2. Let \( P \) be a nonsingular (simple) point on a plane curve \( F \). Show that the tangent line to \( F \) at \( P \) is
   \[ F_x(P)x + F_y(P)y + F_z(P)z = 0. \]

3. Let \( G \) be an irreducible cubic.
   
   (a) Show that \( G \) has at most one singular point, and any such singular point must have multiplicity 2.
   
   (b) Show that if \( G \) has a cusp (i.e., in this case, has a single tangent line of multiplicity 2 at the singular point), then it is projectively equivalent to the curve \( y^2z - x^3 \).
   
   (Hint: Using a problem from the previous problem set, you can start with the singular point at \([0 : 0 : 1]\) and the tangent line \( y \).)

   (c) Show that if \( G \) has a node (i.e., has two distinct tangent lines at the singular point), then it is projectively equivalent to the curve \( xyz = x^3 + y^3 \).

4. (a) Let \( Y \) be a set of 5 distinct points in \( \mathbb{P}^2 \). Let \( V \) be the linear system of conics that contain \( Y \). Show that \( \dim(V) > 0 \) if and only if at least four of the points are collinear.

   (b) Let \( Z \) be a set of 10 distinct points in \( \mathbb{P}^2 \). Let \( W \) be the (possibly empty!) linear system of cubics that contain \( Z \). Show that \( \dim(W) > 0 \) if and only if at least 6 of the points are collinear or at least 9 of the points lie on a conic.

5. Let \( F \) be an irreducible plane curve of degree \( d \). Assume the partial derivative \( F_x \neq 0 \).
   
   (a) If \( P \) is a point on \( F \), show that \( m_P(F_x) \geq m_P(F) - 1 \).

   (b) Using part (b) along with Bézout’s theorem, show that
   \[ \sum_{P \in \mathbb{V}(F)} m_P(m_P - 1) \leq d(d - 1). \]

   (We’ll show in class that this is actually not the best possible bound.)

   (c) Conclude that \( F \) has at most \( \frac{1}{2}d(d - 1) \) multiple points.