For this problem set, you may assume $k$ is algebraically closed.

1. (a) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Show that $I$ is radical if and only if it is equal to the intersection of all the maximal ideals containing it.

(b) Show that the radical of the ideal $I = (x^2 - 2xy^4 + y^6, y^3 - y) \subset \mathbb{C}[x, y]$ is the intersection of three maximal ideals.

2. Let $X = V(x^2 - yz, xz - x) \subset \mathbb{A}_k^3$. Find the irreducible components of $X$ and their corresponding prime ideals.

3. Let $R \subset S \subset T$ be integral domains. If $T$ is integral over $S$ and $S$ is integral over $R$, show that $T$ is integral over $R$.

4. Give an example of rings $R \subset S$ where $S$ is integral but not “ring-finite” over $R$ (where ring-finite means finitely generated as an $R$-algebra).

5. Recall that the dimension of an algebraic set $X \subset \mathbb{A}_k^n$ is the maximum $d$ where $\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d \subseteq X$ and each $X_i$ is an irreducible algebraic set. (In this problem, you can use the following claim, without proof: If $X$ is irreducible, every such sequence of irreducible sets that is not a proper subsequence of another has the same length. This is a consequence of Noether’s Normalization Theorem.)

(a) If $X = V(I)$ give an equivalent definition of dimension in terms of $I$ and $k[x_1, \ldots, x_n]$ and show the two definitions are equivalent.

(b) If $X \subset Y$ are algebraic sets, the codimension of $X$ in $Y$ is $\dim(Y) - \dim(X)$.

Let $I = (f_1, f_2, \ldots, f_c)$. Show that the codimension of $V(I)$ in $\mathbb{A}_k^n$ is at most $c$.

(c) Show that $\dim(\mathbb{A}_k^n) = n$.

6. A subset of affine space $U \subset \mathbb{A}_k^n$ is called compact (in the Zariski topology) if for every collection $\{U_i\}_{i \in J}$ (where $J$ is some indexing set) of Zariski open sets such that

\[ U \subset \bigcup_{i \in J} U_i, \]

then $U$ is also contained in some finite union of the $U_i$. That is, there is some finite set $L \subset J$ such that

\[ U \subset \bigcup_{i \in L} U_i. \]

More concisely, $U$ is compact if every open cover has a finite subcover. Show that if $X \subset \mathbb{A}_k^n$ is an algebraic set, $X$ is compact in the Zariski topology.