As always, assume all rings are commutative, and $k$ is a field.

1. Let $X \subset A^n$ be a set. The **Zariski closure** of $X$, denoted $\overline{X}$, is the intersection of all Zariski closed sets containing $X$. Show that $V(I(X)) = \overline{X}$.

2. A ring $R$ is **reduced** if for all $f \in R$ and $n \in \mathbb{N}$,

   $$f^n = 0 \iff f = 0.$$ 

   Let $I \subset R$ be an ideal. Show that $R/I$ is reduced if and only if $I$ is a radical ideal.

3. Let $I$ be an ideal in a ring $R$. Show that there is a one-to-one correspondence between radical ideals in $R$ containing $I$ and radical ideals in $R/I$.

4. Let $a_1, a_2, \ldots, a_n \in k$. Show that $(x_1 - a_1, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n]$ is a maximal ideal. (Hint: reduce to the case where the $a_i$ are all 0.)

5. Show that a ring $R$ is Noetherian if and only if every strictly increasing sequence of ideals $I_1 \subsetneq I_2 \subsetneq \ldots$ is finite.

6. Let $k = \mathbb{R}$.

   (a) Show that $I(V(x^2 + y^2 + 1)) = (1)$.

   (b) Show that every algebraic set in $\mathbb{A}_\mathbb{R}^2$ is equal to $V(f)$ for some $f \in \mathbb{R}[x, y]$.

7. Assume $k$ is infinite.

   (a) Show $I(\mathbb{A}_k^n) = (0)$.

   (b) Show $\mathbb{A}_k^n$ is irreducible.

   (c) Show that neither (a) nor (b) holds if $k$ is finite.