

VERDIER DUALITY

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1. INTRODUCTION

Let M be a smooth, compact oriented manifold of dimension n , and let k be a field. Recall that there is a natural pairing in singular cohomology

$$H^r(M; k) \times H^{n-r}(M; k) \rightarrow k$$

obtained by the cup-product and an orientation map $H^n(M; k) \simeq k$. Classical Poincaré duality states that this is a perfect pairing, and in particular there is an isomorphism

$$H^r(M; k) \simeq (H^{n-r}(M; k))^\vee.$$

Such a statement is patently false for spaces that are not manifolds, and it is not obvious how to generalize it. The approach to doing so replaces ordinary singular cohomology with *sheaf* cohomology; for a constant sheaf on a manifold, this makes no difference. By abuse of notation, we let k denote the constant sheaf on M with value k , which we also write as \mathcal{F} . In this case, we may write

$$H^r(M; k) = \text{Ext}^r(k, \mathcal{F}), \quad H^{n-r}(M; k) = \text{Ext}^{n-r}(\mathcal{F}, k),$$

and Poincaré duality states equivalently that the *Yoneda product*

$$(1) \quad \text{Ext}^r(k, \mathcal{F}) \times \text{Ext}^{n-r}(\mathcal{F}, k) \rightarrow \text{Ext}^n(k, k) \simeq k$$

is a perfect pairing. Here the Ext groups are calculated in the category of sheaves of k -modules. This statement is actually true for *any* \mathcal{F} , and it is a statement that, while not true in general, is much closer to being generalizable.

These are a collection of expository notes on Verdier duality. Verdier duality is a generalization of Poincaré duality to more general spaces; the caveat is that instead of a simple pairing on cohomology, one has a pairing in the *derived category*, or more generally the existence of an adjoint to the derived push-forward. (This pairing is generally more complicated than (1), but is in the same flavor.) This complication, which seems to arise because the dualizing object on a general space is not very simple (for a manifold, it is cohomologically concentrated in one degree) makes derived categories integral to the statement. The situation is analogous in algebraic geometry, where the generalization (either to the relative case or to stranger schemes) of classical Serre duality for smooth projective varieties becomes an adjunction on the derived category.

These notes also contain classical results on sheaf theory, mostly following [Ive86]. Since I learned basic sheaf theory from Hartshorne's *Algebraic Geometry*, I was not already familiar with lower shrieks and base change when I started reading about Verdier duality, and these results are included (for my own benefit). However, the basics of derived categories as in [GM03] or [Har66] are assumed.

For a space X , we let $\mathbf{Sh}(X)$ be the category of sheaves of abelian groups. More generally, if k is a ring, we let $\mathbf{Sh}(X, k)$ be the category of sheaves of k -modules.

1.1. Preliminaries. Consider a map $f : X \rightarrow Y$ of locally compact spaces. There is induced a push-forward functor $f_* : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$, that sends a sheaf \mathcal{F} (of abelian groups) on X to the push-forward $f_*\mathcal{F}$ on Y . It is well-known that this functor admits a left adjoint f^* , which can be geometrically described in terms of the espace étale as follows: if $\mathfrak{Y} \rightarrow Y$ is the espace étale of a sheaf \mathcal{G} on Y , then $\mathfrak{Y} \times_Y X \rightarrow X$ is the espace étale of $f^*\mathcal{G}$.

Now, under some situations, the functor i_* is very well-behaved. For instance, if $i : Z \rightarrow Y$ is the inclusion of a closed subset, then i_* is an exact functor. It turns out that it admits a *right* adjoint $i^! : \mathbf{Sh}(Y) \rightarrow \mathbf{Sh}(Z)$. This functor can be described as follows.

Let \mathcal{G} be a sheaf on Y . We can define $i^!\mathcal{G} \in \mathbf{Sh}(Z)$ by saying that if $U \subset Z$ is an open subset, such that $U = V \cap Z$ for $V \subset Y$ open, then $i^!(\mathcal{G})(U)$ is the subset of the sections of $\mathcal{G}(V)$ with support in Z . One can check that this does not depend on the choice of open subset V .

Proposition 1.1. *One has the adjoint relation:*

$$\mathrm{Hom}_{\mathbf{Sh}(Z)}(\mathcal{F}, i^!\mathcal{G}) = \mathrm{Hom}_{\mathbf{Sh}(Y)}(i_*\mathcal{F}, \mathcal{G}).$$

Proof. Suppose given a map $i_*\mathcal{F} \rightarrow \mathcal{G}$; we need to produce a map $\mathcal{F} \rightarrow i^!\mathcal{G}$. Indeed, we are given that for each $V \subset Y$ open, there is a map

$$\mathcal{F}(V \cap Z) \rightarrow \mathcal{G}(V).$$

It is easy to see (because $i_*\mathcal{F}$ has support in Z) that only sections of \mathcal{G} with support in Z are in the image. As a result, if $U \subset Z$ is any open set, say $U = V \cap Z$, then we can produce the required map

$$\mathcal{F}(U) \rightarrow i^!\mathcal{G}(U) \subset \mathcal{G}(V).$$

In the reverse direction, if we are given maps $\mathcal{F}(U) \rightarrow \mathcal{G}(V)$ as above, it is easy to produce a map $i_*\mathcal{F} \rightarrow \mathcal{G}$. From this the result follows, as the two constructions are inverse to each other. \square

1.2. Verdier duality. In general, unlike the case of a closed immersion, the functor $f_* : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$ induced by a morphism $f : X \rightarrow Y$ does not have a right adjoint. However, in many nice situations it *will* have a right adjoint on the derived category. Namely, if k is a noetherian ring, let $\mathbf{D}^+(X, k), \mathbf{D}^+(Y, k)$ be the bounded-below derived categories of sheaves of k -modules on X, Y . Before this we had worked with $k = \mathbb{Z}$, but there is no need to avoid this bit of generality.

Since these categories have enough injectives, the functor f_* induces a *derived functor*

$$\mathbf{R}f_* : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Y, k).$$

To compute $\mathbf{R}f_*$ on a bounded-below complex of sheaves \mathcal{F}^\bullet , one finds a quasi-isomorphism

$$\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet,$$

where \mathcal{I}^\bullet is a complex of injective sheaves (of k -modules), and lets $\mathbf{R}f_*\mathcal{F}^\bullet = f_*\mathcal{I}^\bullet$. One checks easily that this is *independent* of the choice of \mathcal{I}^\bullet .

Theorem 1.2 (Verdier duality, first form). *If $f : X \rightarrow Y$ is a proper map of manifolds, then the derived functor $\mathbf{R}f_* : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Y, k)$ admits a right adjoint $f^! : \mathbf{D}^+(Y, k) \rightarrow \mathbf{D}^+(X, k)$.*

We will actually state and prove this fact in more generality later. For now, let us consider a special case: when $Y = *$ is a point. In this case, the derived functor $\mathbf{R}f_*$ is just $\mathbf{R}\Gamma$, in other words the derived functor of the global sections (a.k.a. sheaf cohomology on the derived category). $\mathbf{D}^+(Y, k)$ is the derived category $\mathbf{D}^+(k)$ of the category of k -modules. The result states that there is an adjoint $f^! : \mathbf{D}^+(k) \rightarrow \mathbf{D}^+(X, k)$. Consider the complex with k in degree zero and nothing anywhere else; then let $\mathcal{D}^\bullet \in \mathbf{D}^+(X, k)$ be $f^!(k)$.

By the same result, it follows then that we have an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^+(X,k)}(\mathcal{F}^\bullet, \mathcal{D}^\bullet) \simeq \mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma(\mathcal{F}^\bullet), k).$$

When k is a field, we know that any object in $\mathbf{D}^+(k)$ is isomorphic to its homology; in particular, we find that

$$\mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma(\mathcal{F}^\bullet), k) = (H^0(\mathbf{R}\Gamma(\mathcal{F}^\bullet)))^\vee.$$

For \mathcal{F} simply a sheaf, this last term is $(H^0(X, \mathcal{F}))^\vee$. The complex \mathcal{D} is called a **dualizing complex**. It represents the functor $(H^0(X, \mathcal{F}))^\vee$ on the derived category.

1.3. The statement for non-proper maps. It will be convenient to have a statement of Verdier duality that holds for non-proper maps as well. To do this, given a map of spaces

$$f : X \rightarrow Y,$$

we shall define a new functor

$$f_! : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y).$$

This will coincide with f_* when f is proper. Namely, if $\mathcal{F} \in \mathbf{Sh}(X)$ and $U \subset Y$ is open, we let

$$f_!(\mathcal{F})(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \text{ such that } \mathrm{supp}(s) \text{ is proper over } U\}.$$

We will describe the properties of $f_!$ below, but in any event one sees that there is induced a derived functor $\mathbf{R}f_! : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Y, k)$.

The more general form of Verdier duality will state that $\mathbf{R}f_!$ has a right adjoint (even if f is not proper).

Theorem 1.3. *If X, Y are locally compact of finite dimension,¹ then $\mathbf{R}f_!$ admits a right adjoint $f^! : \mathbf{D}^+(Y, k) \rightarrow \mathbf{D}^+(X, k)$.*

We will prove this theorem in §4 below. The main point of the argument is that while $f_!$ does not have a right adjoint in general, the rather close functor $f_!(\cdot \otimes \mathcal{L})$ does where \mathcal{L} is a suitable (i.e. soft) sheaf. By replacing the constant sheaf by a *complex* of such suitable sheaves, we will be able to define the functor $f^!$ directly on the derived category. In the case of a manifold, we will be able to compute it and recover Poincaré duality.

We will also use the upper shriek functor to define a *dualizing complex* in $\mathbf{D}^+(X, k)$, which will be the image of the constant sheaf on a point. As a result of this, we shall obtain an anti-involution of a subcategory of $\mathbf{D}^+(X, k)$ (the bounded constructible category) on nice spaces.

Remark. (1) Theorem 1.3 is actually true under weaker hypotheses; see [KS02], where it is proved for maps whose fibers are of bounded cohomological dimension.

(2) The existence of the upper shriek can probably be proved by appealing to the Brown representability theorem [Nee01]. Namely, the Brown representability theorem states that a triangulated functor between suitably nice triangulated categories admitting arbitrary direct sum that commutes with direct sums is a left adjoint. However, now one hits a snag: the bounded-below derived categories we use do *not* admit arbitrary resolutions. Fortunately, under finite-dimensionality hypotheses, it is no trouble to define $\mathbf{R}f_!$ on bounded derived categories (because $\mathbf{R}f_!$ will have finite cohomological dimension).

However, I don't see how it is trivial that $\mathbf{R}f_!$ should preserve arbitrary direct sums anyway (before one knows it admits a right adjoint!).

¹Here “finite dimension” is a cohomological condition to be described below.

- (3) There is an analogous theory of duality in étale (or l -adic) cohomology. We briefly describe the former case. Namely, one shows that, for a separated morphism of finite type $f : X \rightarrow S$, the derived push-forward $\mathbf{R}f_*$ admits a right adjoint $f^!$ on the bounded-below derived category of \mathbb{Z}/n -sheaves for some n .

Amazingly, when f is smooth, $f^!$ turns out to behave exactly as one expects from the *topological* case (when the torsion is prime to the characteristic). See [FK88]. For instance, for a smooth variety over an algebraically closed field, the dualizing complex is concentrated in one dimension (as we will show in the topological case in Theorem 5.5), which is essentially the étale version of Poincaré duality.

2. SOFT SHEAVES AND COMPACTLY SUPPORTED COHOMOLOGY

2.1. Softness. Let X be a locally compact space. If $\mathcal{F} \in \mathbf{Sh}(X)$ and $Z \subset X$ is closed, we define $\Gamma(Z, \mathcal{F}) = \Gamma(Z, i^* \mathcal{F})$ for $i : Z \rightarrow X$ the inclusion. This allows us to make sense of a section “over a closed subset.” The interpretation via the espace étale is helpful here: if $\mathfrak{X} \rightarrow X$ is the espace étale of \mathcal{F} , then a section over Z is the same thing as a section of the projection $\mathfrak{X} \rightarrow X$ over Z . It follows quite cleanly from the latter interpretation that if we have sections s, t on closed subsets F, G that agree on $F \cap G$, one obtains uniquely a closed subset on $F \cup G$ (though of course one could argue directly).

Now we want to show that we can recover this notion from the familiar idea of sections over an *open* set.

Lemma 2.1. *If $Z \subset X$ is compact, we have*

$$\Gamma(Z, \mathcal{F}) \simeq \varinjlim \Gamma(U, \mathcal{F}),$$

where U ranges over open sets containing Z .

Proof. There is an obvious map $\varinjlim \Gamma(U, \mathcal{F}) \rightarrow \Gamma(Z, \mathcal{F})$. It is injective: if $s \in \Gamma(U, \mathcal{F})$ restricts to zero over Z , then the stalk $s_z, z \in Z$ is zero. It follows that there is a neighborhood $U_z \subset U$ for each $z \in Z$ such that $s|_{U_z} = 0$. But then $s|_{\bigcup U_z} = 0$, so that s maps to zero in the colimit. (We have not yet used the compactness of Z .)

Surjectivity is the hard part. Suppose $t \in \Gamma(Z, \mathcal{F})$ is a section over the compact set Z . Then we can cover Z by neighborhoods $\{N_\alpha\}$ in X together with sections $s_\alpha \in \mathcal{F}(N_\alpha)$ such that

$$s_\alpha|_{Z \cap N_\alpha} = t|_{Z \cap N_\alpha}.$$

By choosing the N_α appropriately and in view of local compactness, we can assume that there are *compact* subsets $Y_\alpha \subset N_\alpha$, each of which contains an open subset of Z , and whose interiors (with respect to Z !) cover Z . It follows that a *finite* number of the $\{Y_\alpha\}$ cover Z .

So we are in the following situation. We have a section $t \in \Gamma(Z, \mathcal{F})$ which we want to extend to an open neighborhood; a finite set Y_1, \dots, Y_k of closed subsets of X that cover Z , and open sets N_1, \dots, N_k with sections $s_i \in \mathcal{F}(N_i)$ that agree with t on Y_i . We want to find an s defined over some open neighborhood of Z that looks locally like the s_i , near Z .

To do this, we can assume inductively that $k = 2$, extending piece by piece. So then we have two compact subsets Y_1, Y_2 , and sections s_1, s_2 defined in a neighborhood of each. We are given that $s_1|_{Y_1 \cap Y_2} = s_2|_{Y_1 \cap Y_2}$; we now wish to find a section defined on a neighborhood of $Y_1 \cup Y_2$. To do this, choose a small neighborhood N of $Y_1 \cap Y_2$ such that $s_1|_N = s_2|_N$, which we can do by the first part of the proof. Then choose disjoint neighborhoods N_1, N_2 such that $N_1 \cup N$ contains Y_1 , and $N_2 \cup N$ contains Y_2 . It is easy to see that $s_1|_{N_1}, s_1|_N, s_2|_{N_2}$ all patch appropriately and give the desired section. \square

We shall need the following weaker form of flasqueness.

Definition 2.2. A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is **soft** if $\mathcal{F}(X) \rightarrow \mathcal{F}(Z)$ is surjective whenever $Z \subset X$ is compact.

By Theorem 2.1, a flasque sheaf is readily seen to be soft. We note that the restriction of a soft sheaf to a closed (or open!) subset is always still soft, as follows easily from the definition.

We have the following property of soft sheaves:

Proposition 2.3. *A sheaf \mathcal{F} on a locally compact space X is soft if and only if any section of \mathcal{F} over a compact subset $K \subset X$ can be extended to a compactly supported global section of X .*

Even more precisely, if \mathcal{F} is soft, $K \subset U$ is an inclusion of a compact set K in an open set U , then a section over K can be extended to a compactly supported global section with support in U .

Proof. The definition of softness is that any section over K (K compact but arbitrary) can be extended to X . So one direction is clear. We have to show conversely that if \mathcal{F} is soft, and $s \in \Gamma(K, \mathcal{F})$ is a section over the compact subset K , then we can choose the extending global section in such a way that it is compactly supported (and that we can take the support inside U).

To do this, choose a compact set L containing K in its interior, and with $L \subset U$. Consider the section on $K \cup \partial L$ given by s on K and zero on ∂L . By softness, this can be extended to a section t of \mathcal{F} over L that vanishes at the boundary. Now the sections given by t on L and 0 on $\overline{X - L}$ glue and give a compactly supported extension of s ; it is clear that it vanishes outside U . \square

We have seen that flasque sheaves are soft; in particular, injective sheaves are soft, because injective sheaves are flasque. But soft sheaves are more general than flasque sheaves. We illustrate this below.

Example. Let X be a locally compact space, and suppose \mathcal{C} is the sheaf of continuous functions on X . Then \mathcal{C} is a soft sheaf. Indeed, suppose given a section s of \mathcal{C} over a compact set K , i.e. a continuous function defined in some neighborhood U of K . By multiplying s by a *globally defined* continuous function t supported in U such that $t|_K \equiv 1$ (Urysohn's lemma!), we get a section of $\Gamma(U, \mathcal{C})$ which vanishes outside a compact set. We can thus extend this to a global continuous function.

Example. Similarly, using suitable cutoff functions, it follows that if M is a manifold, then the sheaf of C^p functions (where $p \in \mathbb{N} \cup \{\infty\}$) is soft.

Example. If \mathcal{A} is a soft sheaf of rings, and \mathcal{F} is a sheaf of \mathcal{A} -modules, then \mathcal{F} is a flasque sheaf. Indeed, let $s \in \Gamma(K, \mathcal{F})$ be a section. Then s comes from a section of \mathcal{F} over some open subset U containing K . We can find a global section $i \in \Gamma(X, \mathcal{A})$ with support in U that is identically 1 on K by softness. (Indeed, this follows from Theorem 2.3.) Then si extends to a global section extending s on K .

It follows in particular that if M is a locally compact space and \mathcal{E} is any vector bundle, the sheaf of sections of \mathcal{E} is a soft sheaf, as it is a sheaf of modules over the sheaf of continuous functions. If M is a manifold and \mathcal{E} a smooth vector bundle, the same holds for the sheaf of smooth sections. For instance, the sheaves of k -forms are all soft.

The final example will be useful to show that one can compute the cohomology of a smooth manifold using the de Rham resolution $0 \rightarrow \mathbb{R} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \dots$, where Ω^i is the sheaf of smooth i -forms and the coboundary maps are given by exterior differentiation.

2.2. Compactly supported cohomology.

Let X be a topological space.

Definition 2.4. If $\mathcal{F} \in \mathbf{Sh}(X)$, and $U \subset X$ is open, we define $\Gamma_c(U, \mathcal{F})$ to be the subgroup of $\Gamma(U, \mathcal{F})$ consisting of sections with compact support.

It is easy to see that if $s, t \in \Gamma(U, \mathcal{F})$ have compact support, so does $s + t$. So this is indeed a subgroup.

Taking $U = X$, we thus get a functor $\Gamma_c = \Gamma_c(X, \cdot) : \mathbf{Sh}(X) \rightarrow \mathbf{Ab}$ for \mathbf{Ab} the category of abelian groups. Note similarly that Γ_c takes $\mathbf{Sh}(X, k)$ to the category of k -modules.

Proposition 2.5. Γ_c is left-exact.

Proof. Indeed, we know that the ordinary global section functor Γ is left-exact. Since $\Gamma_c \subset \Gamma$, all that needs to be seen is that if one has a sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$, and $s \in \Gamma_c(X, \mathcal{F})$ maps to zero in $\Gamma_c(X, \mathcal{F}'')$, then s comes from a *compactly supported* section in $\Gamma(X, \mathcal{F}')$. But it comes from *some* (unique!) section of \mathcal{F}' , and injectivity of $\mathcal{F}' \rightarrow \mathcal{F}$ shows that the section must be compactly supported. \square

Definition 2.6. We write $H_c^i(X, \cdot)$ for the i th (right) derived functor of $\Gamma_c(X, \cdot)$. We can also define a *total derived functor* $\mathbf{R}\Gamma_c$ on $\mathbf{D}^+(X)$, taking values in the (bounded-below) derived category of abelian groups. More generally there is a functor $\mathbf{R}\Gamma_c : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(k)$ for $\mathbf{D}^+(k)$ the derived category of k -modules.

In the classical theory of sheaf cohomology, flasque sheaves (recall that a sheaf \mathcal{F} is *flasque* if $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective for each open set) are the prototypical sheaves with no cohomology. For our purposes, it will be convenient to use the less restrictive notion of a *soft* sheaf, because for instance the de Rham resolution (see ?? below) is a soft but not flasque resolution.

We will show this using:

Proposition 2.7. Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of sheaves on X . Suppose \mathcal{F}' is soft. Then the sequence $0 \rightarrow \Gamma_c(X, \mathcal{F}') \rightarrow \Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{F}'') \rightarrow 0$ is also exact.

Proof. We only need to prove surjectivity at the end.

We shall start by making a reduction. In view of Theorem 2.3, we can assume that X is itself compact. Indeed, let $s \in \mathcal{F}''(X)$ have compact support Z . Choose a compact subset $Z' \subset X$ containing Z in its interior. Then by applying the result (assumed for compact spaces) to Z' , we get a section $s' \in \Gamma(Z', \mathcal{F})$ lifting s . Then $s'|_{\partial Z'}$ must live inside $\Gamma(\partial Z', \mathcal{F}')$ as it maps to zero in \mathcal{F}'' , and this restriction can be extended to some global section $t \in \Gamma(X, \mathcal{F}')$. It follows that $s' - t$ is a section of \mathcal{F} over Z' that restricts to zero on the boundary, so it extends to a section of \mathcal{F} over X .

With this reduction made, let us assume X compact. In this case, compactly supported global sections are just the same thing as ordinary global sections. So X is a compact space, and there is thus a *finite* cover of X by compact sets $V_i, 1 \leq i \leq n$, such that there exist liftings $t_i \in \mathcal{F}(V_i)$ of $s|_{V_i}$. We can choose these compact sets V_i , moreover, such that their *interiors* cover X . This follows easily from the surjectivity of $\mathcal{F} \rightarrow \mathcal{F}''$. Now we need to piece together the t_i . To do this, we reduce by induction to the case $n = 2$, and there are two sections to piece together.

Consider the sections t_1, t_2 . If they agreed on $V_1 \cap V_2$, then we would be done. In general, they need not; however, $t_1|_{V_1 \cap V_2} - t_2|_{V_1 \cap V_2}$ is necessarily a section e_1 of $\mathcal{F}'(V_1 \cap V_2)$ as it maps to zero in $\mathcal{F}''(V_1 \cap V_2)$. By softness, we can extend e_1 to a global section e of \mathcal{F}' . Then it is clear that the pair $t_1, t_2 + e$ of sections of \mathcal{F} glues to give a global section. \square

We can now derive two useful properties of soft sheaves.

Corollary 2.8. (1) If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence in $\mathbf{Sh}(X)$ and $\mathcal{F}', \mathcal{F}$ are soft, then \mathcal{F}'' is soft too.

(2) $H_c^i(X, \mathcal{F}) = 0$ for $i > 1$ if \mathcal{F} is soft.

Proof. (1) Let $Z \subset X$ be a closed subset. We know by Theorem 2.7 that $\Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{F}'')$ is surjective. Similarly, $\Gamma_c(Z, \mathcal{F}) \rightarrow \Gamma_c(Z, \mathcal{F}'')$ is surjective. In the diagram

$$\begin{array}{ccc} \Gamma_c(X, \mathcal{F}) & \longrightarrow & \Gamma_c(X, \mathcal{F}'') \\ \downarrow & & \downarrow \\ \Gamma_c(Z, \mathcal{F}) & \longrightarrow & \Gamma_c(Z, \mathcal{F}''), \end{array}$$

we see that the horizontal maps and the left vertical map are all surjective. It follows that $\Gamma_c(X, \mathcal{F}'') \rightarrow \Gamma_c(Z, \mathcal{F}'')$ is surjective, which implies softness.

(2) This now follows by a general lemma on acyclicity, proved in [Gro57]. We can just repeat it for the special case of interest. Let us first prove that $H_c^1(X, \mathcal{F}) = 0$ for \mathcal{F} soft. To do this, imbed \mathcal{F} in an injective sheaf \mathcal{I} ; the cokernel \mathcal{G} is soft by the previous part of the result. The exact sequence

$$0 \rightarrow \Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{I}) \rightarrow \Gamma_c(X, \mathcal{G}) \rightarrow H_c^1(X, \mathcal{F}) \rightarrow H_c^1(X, \mathcal{I}) = 0$$

shows that $H_c^1(X, \mathcal{F}) = 0$ for *any* soft sheaf \mathcal{F} .

Now assume inductively that $H_c^n(X, \mathcal{F}) = 0$ for any soft sheaf \mathcal{F} . Then the isomorphisms $H_c^n(X, \mathcal{G}) \simeq H_c^{n+1}(X, \mathcal{F})$ and the inductive hypothesis applied to \mathcal{G} (which is also soft, as it is injective and hence flasque) complete the inductive step. □

2.3. The $f_!$ functors. Let $f : X \rightarrow Y$ be a map of spaces. We have defined the functor

$$f_! : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$$

earlier, such that $f_!(U)$ consists of the sections of $\mathcal{F}(f^{-1}(U))$ whose support is proper over U ; $f_!\mathcal{F}$ is always a subsheaf of $f_*\mathcal{F}$, equal to it if f is proper. When Y is a point, we get the functor

$$\mathcal{F} \mapsto \Gamma_c(X, \mathcal{F}) = \{\text{global sections with proper support}\}.$$

One can check that $f_!\mathcal{F}$ is in fact a sheaf. The observation here is that a map $A \rightarrow B$ of topological spaces is proper if and only if there is an open cover $\{B_i\}$ of B such that $A \times_B B_i \rightarrow B_i$ is proper for each i .

Now $f_!$ is a left-exact functor, as one easily sees. We now want to show that the class of soft sheaves is acyclic with respect to $f_!$, and in particular that one may use soft resolutions to compute the derived functors. To do this, we shall prove a general “base change” theorem that will compute the stalk of $f_!\mathcal{F}$.

Definition 2.9. As usual, $R^i f_!$ denotes the i th right derived functor of $f_!$. There is also a total derived functor on the derived categories $\mathbf{R}f_! : \mathbf{D}^+(X) \rightarrow \mathbf{D}^+(Y)$ (and similarly $\mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Y, k)$).

The point of Verdier duality is that, while $f_!$ is usually not a left adjoint, $\mathbf{R}f_!$ will be (under decent circumstances).

2.4. Base change theorems. To understand the functor $f_!$ and its derived functors $R^i f_!$, we will need to determine their stalks. We start by describing the situation for a *proper* map.

Theorem 2.10 (Proper base change). *Let $f : X \rightarrow Y$ be a proper map of locally compact spaces. If $\mathcal{F} \in \mathbf{Sh}(X)$, then there is a functorial isomorphism*

$$(R^i f_* \mathcal{F})_y \simeq H^i(X_y, \mathcal{F}_y)$$

for each $y \in Y$.

Here \mathcal{F}_y is the restriction of \mathcal{F} to X_y , which in turn is the fiber over y .

Proof. Note that both $(R^i f_* \mathcal{F})_y, H^i(X_y, \mathcal{F}_y)$ are δ -functors from $\mathbf{Sh}(X)$ to the category of abelian groups.

Let us first define the natural transformation. Indeed, we know that $R^i f_* \mathcal{F}$ is the sheaf on Y associated to the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

As a result, the stalk at y is the direct limit $\varinjlim_U H^i(f^{-1}(U), \mathcal{F})$, where U ranges over open sets containing y . Each $H^i(f^{-1}(U), \mathcal{F})$ maps naturally to $H^i(X_y, \mathcal{F}_y)$, so we get the natural map (of δ -functors, even).

Let us show that it is an isomorphism when $i = 0$. This equates to saying that the map

$$\varinjlim \mathcal{F}(f^{-1}(U)) \rightarrow \Gamma(X_y, \mathcal{F}|_{X_y})$$

is an isomorphism. But since f is a closed map, and X_y is compact, it follows that the sets $f^{-1}(U)$ for U open and containing y form a cofinal family in the set of open sets containing X_y . The claim is now clear from Theorem 2.1.

Finally, we need to show that both functors are effaceable (in the terminology of [?]) in positive dimensions. For $(R^i f_*)_y$, this is immediate. For the other functor, let us show that if \mathcal{F} is soft, then $H^i(X_y, \mathcal{F}_y) = 0$ for $i > 0$. Since any sheaf can be imbedded in a soft (e.g. flasque) sheaf, this will be enough. But this in turn is clear because \mathcal{F}_y is itself then soft:

Lemma 2.11. *The restriction of a soft sheaf to a locally closed subspace is soft.*

Proof. Indeed, this follows because if $\mathcal{F} \in \mathbf{Sh}(X)$ is soft and $Z \subset X$, then any compact subset of Z is a compact subset of X . So a section of \mathcal{F} over Z can be extended all the way over X , and a fortiori over Z . \square

\square

Here is the analog for the $f_!$ functor:

Theorem 2.12. *Let $f : X \rightarrow Y$ be a continuous map of locally compact spaces. If $\mathcal{F} \in \mathbf{Sh}(X)$, then there is a functorial isomorphism*

$$(R^i f_! \mathcal{F})_y \simeq H_c^i(X_y, \mathcal{F}_y)$$

for each $y \in Y$.

Proof. This is now proved as before. Again, the key point is that a map of δ -functors can easily be defined, which is an isomorphism in degree zero, and both functors are effaceable in positive degrees.

But there are some subtleties! For one thing, it is *not* true that $R^i f_! \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H_c^i(U, \mathcal{F})$. This fails even if $i = 0$ (take f the identity map, for instance). However, by general nonsense, if we can define an isomorphism in degree zero, and show that both functors are effaceable, then we will be done. Both functors are effaceable since $R^i f_!$ is effaceable (being a derived functor) while $\mathcal{F} \mapsto H_c^i(X_y, \mathcal{F}_y)$ vanishes for soft sheaves.

So we just have to check that $(f_! \mathcal{F})_y = \Gamma_c(X_y, \mathcal{F}_y)$. Now there is a natural $(f_! \mathcal{F})_y \rightarrow \Gamma_c(X_y, \mathcal{F}_y)$ that sends a section of $\mathcal{F}(f^{-1}(U))$, with support proper over U , to the restriction to X_y . This map is injective; for if a section $s \in \mathcal{F}(f^{-1}(U))$ with proper support was zero on $f^{-1}(y)$, then the image of $\text{supp}(s)$ in U would not contain y . But by properness this image is closed, so we can find a smaller neighborhood V containing y such that $\text{supp}(s) \cap f^{-1}(V) = \emptyset$.

Now we need to check surjectivity. This is a bit tricky to do directly, but fortunately a trick (which I learned from [Ive86]) helps out here. We know that $(f_! \mathcal{F})_y \rightarrow \Gamma_c(X_y, \mathcal{F}_y)$ is surjective when \mathcal{F} is soft: this is easy to see (for an element of $\Gamma_c(X_y, \mathcal{F})$ can be extended to all of X with compact, and certainly proper support). Given \mathcal{F} , we find an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{J}$$

with \mathcal{I}, \mathcal{J} soft (e.g. injective). Since the map from the stalk to $\Gamma_c(X_y, \cdot)$ is an isomorphism for \mathcal{I}, \mathcal{J} , and since both functors are obviously left exact, a diagram chase shows that the map is an isomorphism for \mathcal{F} . \square

2.5. Applications. We start by proving a result that provides some content to the phrase “base change.”

Theorem 2.13. *Consider a cartesian diagram of locally compact Hausdorff spaces:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow p' & & \downarrow p \\ S' & \xrightarrow{f} & S \end{array}$$

Then there is a natural isomorphism, for any $\mathcal{F}^\bullet \in \mathbf{D}^+(X)$,

$$f^* \mathbf{R}p_! \mathcal{F}^\bullet \simeq \mathbf{R}p'_! f'^* \mathcal{F}^\bullet.$$

Taking cohomology, one sees that for a sheaf \mathcal{F} , one has natural isomorphisms

$$f^* R^i p_! \mathcal{F} \simeq R^i p'_! f'^* \mathcal{F}.$$

Proof. One may define this map by the universal property, on the level of complexes. Namely, let \mathcal{G}^\bullet be a complex of sheaves. We will define a map

$$f^* p_! \mathcal{G}^\bullet \rightarrow p'_! f'^* \mathcal{G}^\bullet.$$

To do this, we may as well assume \mathcal{G}^\bullet is a single sheaf \mathcal{G} (by naturality). So we are reduced to defining a map

$$f^* p_! \mathcal{G} \rightarrow p'_! f'^* \mathcal{G}, \quad \mathcal{G} \in \mathbf{Sh}(X).$$

This is equivalent to defining a map $p_! \mathcal{G} \rightarrow f_* p'_! f'^* \mathcal{G}$. Given a section of $p_! \mathcal{G}$ over an open set U , or equivalently a section s of $\mathcal{G}(p^{-1}(U))$ with proper support over U , we can consider this as a section of $f^* \mathcal{G}$ over $f'^{-1}(p^{-1}(U)) = p'^{-1}(f^{-1}(U))$ with proper support over $f^{-1}(U)$. This is equivalently a section of $p'_! f'^* \mathcal{G}$ over $f^{-1}(U)$, or a section of $f_* p'_! f'^* \mathcal{G}$ over U . So we can get the base change morphism, which is clearly natural. From here it is easy to see that the base change morphism can be defined even in the derived category, by taking \mathcal{F}^\bullet to be a bounded-below complex of injectives.

To check that it is an isomorphism, we reduce by general facts on “way-out” functors (proved in [Har66]) to showing that the map is an isomorphism for a single sheaf \mathcal{F} , or equivalently that

$$f^* R^i p_! \mathcal{F} \simeq R^i p'_! f'^* \mathcal{F}.$$

But now this follows by taking the stalks at some $s' \in S$; on the left, we get $H_c^i(p^{-1}(f(s)), \mathcal{F}|_{p^{-1}(f(s))})$, and on the right we get $H_c^i(p'^{-1}(s'), f'^*\mathcal{F}|_{p'^{-1}(s')})$, which are both the same since these are isomorphic spaces. (By abuse of notation we have written \mathcal{F} for the restrictions to various subspaces.) \square

As another example of these base change theorems, we prove the promised result that soft sheaves are acyclic with respect to the lower shriek.

Proposition 2.14. *Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of sheaves on X . Suppose \mathcal{F}' is soft. Then the sequence $0 \rightarrow f_!\mathcal{F}' \rightarrow f_!\mathcal{F} \rightarrow f_!\mathcal{F}'' \rightarrow 0$ is also exact, and $\mathbf{R}f_!\mathcal{F}$ is cohomologically concentrated in degree zero (or, what is the same thing, $R^i f_!\mathcal{F}$ vanish for $i > 0$).*

Proof. It follows that the sequence $0 \rightarrow f_!\mathcal{F}' \rightarrow f_!\mathcal{F} \rightarrow f_!\mathcal{F}'' \rightarrow 0$ is exact if $R^1 f_!\mathcal{F}' = 0$. But more generally $R^i f_!\mathcal{F}' = 0$ because taking stalks at each $y \in Y$ gives $H_c^i(X_y, \mathcal{F}') = 0$. \square

Thus, *soft* resolutions will suffice to compute $\mathbf{R}f_!$, which will be very convenient.

As another example of base change, consider an open immersion $j : U \rightarrow X$. By looking at stalks (and defining a natural map), we find that $j_!\mathcal{F}$ for $\mathcal{F} \in \mathbf{Sh}(U)$ is just “extension by zero.”

2.6. The Leray spectral sequence for $\mathbf{R}f_!$. In fact, $f_!$ even preserves soft sheaves, which leads to a Leray spectral sequence for $\mathbf{R}f_!$.

Proposition 2.15. *If $f : X \rightarrow Y$ is continuous and $\mathcal{F} \in \mathbf{Sh}(X)$ soft, then $f_!\mathcal{F}$ is soft too.*

Proof. Indeed, let $Z \subset Y$ be a compact set, and suppose $s \in \Gamma(Z, f_!\mathcal{F})$ is a section, so we know that s extends to a small neighborhood U of Z ; call the extension \tilde{s} . Then this extension \tilde{s} of s becomes a section of $f^{-1}(U)$ with proper support. Restricting U to a *compact* neighborhood of Z , we get a compactly supported section of \mathcal{F} , which we can extend to all of X so as to have compact support (and thus get a global section of $f_!\mathcal{F}$). \square

Corollary 2.16 (Leray spectral sequence). *Given maps $f : X \rightarrow Y, g : Y \rightarrow Z$ of locally compact Hausdorff spaces, there is a natural isomorphism $\mathbf{R}(g \circ f)_! \simeq \mathbf{R}g_! \circ \mathbf{R}f_!$, and a spectral sequence for any $\mathcal{F} \in \mathbf{Sh}(X)$,*

$$R^i g_! R^j f_!\mathcal{F} \implies R^{i+j}(g \circ f)_!\mathcal{F}.$$

Proof. This is now clear, because $f_!$ maps injective sheaves (which are flasque, hence soft) to soft sheaves, which are $g_!$ -acyclic, and we can apply the general theorem. \square

In the special case where Z is a point, we get a spectral sequence

$$H_c^i(Y, R^j f_!\mathcal{F}) \implies H_c^{i+j}(X, \mathcal{F}).$$

3. COHOMOLOGICAL DIMENSION

The Verdier duality theorem will apply not only to manifolds, but more generally to locally compact spaces of finite cohomological dimension, and it will thus be useful to show that simple spaces (e.g. finite-dimensional CW complexes) satisfy this condition. The resulting theory will also show that much of basic algebraic topology can be done entirely using sheaf cohomology.

Definition 3.1. A locally compact space X has **cohomological dimension** n if $H_c^k(X, \mathcal{F}) = 0$ for any sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ and $k > n$, and n is the smallest integer with these properties. We shall write $\dim X$ for the cohomological dimension of X .

A point, for instance, has cohomological dimension zero. For here the global section functor is an equivalence of categories between $\mathbf{Sh}(\{*\})$ and the category of abelian groups.

Our first major goal will be to show that any interval in \mathbb{R} has cohomological dimension one. If we had used ordinary (not compactly supported) cohomology, this would follow from the fact that the *topological* dimension of \mathbb{R} is one: namely, one has a cofinal set of coverings of \mathbb{R} such that any intersection of three distinct elements is trivial (e.g. using open intervals). Since Čech cohomology with alternating cochains suffices to compute sheaf cohomology on a paracompact space (see [God98], for instance), it follows that:

Theorem 3.2. $H^n(\mathbb{R}, \mathcal{F}) = 0$ if $n \geq 2$ and $\mathcal{F} \in \mathbf{Sh}(\mathbb{R})$ is any sheaf.

It will be a bit trickier to obtain this for *compactly supported* cohomology. Before this, we shall develop a number of elementary ideas.

3.1. Softness. To start with, we shall characterize cohomological dimension using softness. Throughout, k will be a *noetherian* ring.

Lemma 3.3. A sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ is soft if and only if $H_c^1(U, \mathcal{F}) = 0$ whenever $U \subset X$ is open.

There is an analog of this result for flasque sheaves that uses local cohomology.

Proof. We already know that softness implies this. Conversely, suppose the vanishing hypothesis satisfied. If $Z \subset X$ is any closed set, then there is a short exact sequence

$$0 \rightarrow j_!j^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*i^*\mathcal{F} \rightarrow 0$$

where $j : X - Z \rightarrow X$ is the inclusion, $i : Z \rightarrow X$ is the inclusion, and $j_!$ denotes extension by zero. If we apply Γ_c , we get a piece of the long exact sequence

$$\Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(Z, \mathcal{F}) \rightarrow H_c^1(X, j_!j^*\mathcal{F}).$$

The claim is that $H_c^1(X, j_!j^*\mathcal{F}) \simeq H_c^1(U, j^*\mathcal{F})$. If we have proved this, it will follow that $\Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(Z, \mathcal{F})$ is always surjective, implying softness. But this follows from the Leray spectral sequence

$$H_c^p(X, R^qj_!\mathcal{G}) \implies H_c^{p+q}(U, \mathcal{G})$$

valid for any $\mathcal{G} \in \mathbf{Sh}(U)$, and the fact that $j_!$ (“extension by zero”) is exact. □

We saw as a corollary of the above proof that whenever $Z \subset X$ was closed, $U = X - Z$, then there is a long exact sequence

$$\dots \rightarrow H_c^i(U, \mathcal{F}) \rightarrow H_c^i(X, \mathcal{F}) \rightarrow H_c^i(Z, \mathcal{F}) \rightarrow H_c^{i+1}(U, \mathcal{F}) \rightarrow \dots$$

This long exact sequence does *not* make sense in ordinary cohomology. In fact, there is no reason to expect maps $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F})$ in the first place. This does, however, make perfect sense in compactly supported cohomology, because one simply extends by zero.

Proposition 3.4. Let X be a locally compact space, and $n = \dim X$, then in any sequence

$$(2) \quad 0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_{n+1} \rightarrow 0,$$

if $\mathcal{F}_1, \dots, \mathcal{F}_n$ are all soft, so is \mathcal{F}_{n+1} .

Conversely, if in every exact sequence (2), the softness of $\mathcal{F}_1, \dots, \mathcal{F}_n$ implies that of \mathcal{F}_0 , it follows that $\dim X \leq n$.

Notice that no hypothesis is made on \mathcal{F}_0 .

Proof. We know that each of the $\mathcal{F}_i, 1 \leq i \leq n$, has no (compactly supported) cohomology above dimension one. By a standard “dimension-shifting” argument, it follows that $H_c^1(U, \mathcal{F}_{n+1}) \simeq H_c^{n+1}(U, \mathcal{F}_0) = 0$ for each $U \subset X$ open. This implies softness by Theorem 3.3.

Let us now prove the last claim. If $\mathcal{F} \in \mathbf{Sh}(X)$ is any sheaf, then there is an injective resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$$

We truncate this resolution at the length n ; that is, we consider the sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots \rightarrow \mathcal{I}^{n-1} \rightarrow \text{im}(\mathcal{I}^{n-1} \rightarrow \mathcal{I}^n) \rightarrow 0.$$

By assumption, the last term is soft; it follows that any sheaf has a soft resolution of length at most n . Since soft resolutions can be used to compute compactly supported cohomology, it follows that $H_c^i(X, \mathcal{F}) = 0$ if $i > n$, proving the claim. \square

Corollary 3.5. *If X is a finite-dimensional space and \mathcal{F} is a sheaf admitting a soft resolution from the left, then \mathcal{F} is soft.*

We shall use the following fact in the proof of Verdier duality.

Corollary 3.6. *If X is a finite-dimensional space, $\mathcal{F}, \mathcal{G} \in \mathbf{Sh}(X, k)$, and \mathcal{G} is soft and k -flat, then $\mathcal{F} \otimes_k \mathcal{G}$ is soft as well.*

Here “ k -flat” means that the stalks are flat k -modules.

Proof. Indeed, we can find a resolution of \mathcal{F} ,

$$\dots \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow 0,$$

where each \mathcal{F}_i is a direct sum of sheaves of the form $j_!(k)$ for j an inclusion of an open subset of X . (Indeed, we can use such sheaves to hit each section of \mathcal{F} .) Since \mathcal{G} is k -flat, we get a resolution of $\mathcal{F} \otimes_k \mathcal{G}$,

$$\dots \rightarrow \mathcal{F}_3 \otimes_k \mathcal{G} \rightarrow \mathcal{F}_2 \otimes_k \mathcal{G} \rightarrow \mathcal{F}_1 \otimes_k \mathcal{G} \rightarrow \mathcal{F} \otimes_k \mathcal{G} \rightarrow 0,$$

where each of the $\mathcal{F}_i \otimes_k \mathcal{G}$ is soft. Indeed, any sheaf of the form $j_!(k) \otimes_k \mathcal{G} \simeq j_!(\mathcal{G}|_U)$ (if $j : U \rightarrow X$ is the inclusion) is soft because \mathcal{G} is soft. We saw this in the proof of the Leray spectral sequence for the lower shriek.

Now by Theorem 3.5 the softness of $\mathcal{F} \otimes_k \mathcal{G}$ is clear. \square

This shows that if $k \rightarrow \mathcal{L}^\bullet$ is a soft, flat resolution of the constant sheaf k , then there is a *functorial* soft resolution $\mathcal{F} \rightarrow \mathcal{F} \otimes_k \mathcal{L}^\bullet$ of any $\mathcal{F} \in \mathbf{Sh}(X, k)$. Of course, we should check that k in fact has a soft, flat resolution. To get this, we can use the *Godement resolution*, which is based on an imbedding of a sheaf $\mathcal{F} \in \mathbf{Sh}(X, k)$ into a soft (even flasque) sheaf $C(\mathcal{F}) = \prod_{x \in X} (i_x)_* \mathcal{F}_x$, where $i_x : \{x\} \rightarrow X$ is the inclusion. The Godement resolution of a given sheaf is a quasi-isomorphism

$$\mathcal{F} \rightarrow C^\bullet(\mathcal{F}),$$

where $C^0(\mathcal{F}) = C(\mathcal{F})$, $C^1(\mathcal{F}) = C(\text{coker}(\mathcal{F} \rightarrow C^0(\mathcal{F})))$, and so on.

Lemma 3.7. *The Godement resolution of a flat sheaf is a complex of soft, flat sheaves.*

Proof. It is clear that the Godement resolution consists of soft (even flasque) sheaves because $C(\mathcal{F})$ is *always* flasque. We now need to show that the sheaves are flat. If \mathcal{F} is flat, then $C(\mathcal{F})$ is flat (as a k -sheaf) because, for a noetherian ring, the product (even infinite!) of flat modules is flat. Moreover, because the map $\mathcal{F} \rightarrow C(\mathcal{F})$ is a *split* injection on the level of stalks, the cokernel is also flat, and we can see that $C^\bullet(\mathcal{F})$ is flat by induction. \square

We shall use the Godement resolution of k to get a functorial “soft replacement” in the derived category in the proof of Verdier duality.

3.2. Subspaces. Verdier duality applies to spaces of finite (cohomological) dimension, so we shall need some criteria to establish that a space is indeed of this form. Here we shall handle some of the simplest ones.

We now want to show:

Proposition 3.8. *If $Z \subset X$ is a locally closed subspace, then $\dim Z \leq \dim X$.*

Proof. We need to check this for open subspaces and for closed subspaces. If $Z \subset X$ is closed, then for any $\mathcal{F} \in \mathbf{Sh}(Z)$, we have that

$$H_c^i(Z, \mathcal{F}) \simeq H_c^i(Z, i_*\mathcal{F})$$

for $i : Z \rightarrow X$ the inclusion, as one may see from (for instance) the Leray spectral sequence. Thus the result $\dim Z \leq \dim X$ is clear.

If $U \subset X$ is open with $j : U \rightarrow X$ the inclusion, then we know that

$$\Gamma_c(X, j_!\mathcal{G}) \simeq \Gamma_c(U, \mathcal{G}), \quad \mathcal{G} \in \mathbf{Sh}(U).$$

The corresponding identity on derived functors (which follows from the Leray spectral sequence, as $j_!$ is exact, for instance) yields

$$H_c^i(X, j_!\mathcal{G}) \simeq H_c^i(U, \mathcal{G}),$$

which easily implies that $\dim U \leq \dim X$. □

Nonetheless, dimension is a “local” invariant by the following fact:

Proposition 3.9. *If X is a locally compact space and $\{U_\alpha\}$ a covering of X by open sets, then*

$$\dim X = \sup \dim U_\alpha.$$

The same holds for a finite covering by compact sets.

Proof. This now follows from the above criteria and the technical fact that a sheaf which is locally soft is soft. That is, if $\mathcal{F} \in \mathbf{Sh}(X)$ and there is a cover $\{U_\alpha\}$ of X (which is either open or a finite cover by compact sets) such that $\mathcal{F}|_{U_\alpha}$ is soft for each U_α , then \mathcal{F} is itself soft.

This may be seen as follows. We first treat the compact case, so that X is compact. Let $Z \subset X$ be a compact set and $s \in \Gamma(Z, \mathcal{F})$ be a section. Let U_1, \dots, U_k be compact sets covering X such that $\mathcal{F}|_{U_i}$ is soft for each i . Then we want to show that \mathcal{F} is soft.

We will construct extensions s_i over $Z \cup U_1 \cup \dots \cup U_i$. When $i = 0$, there is nothing to do. If s_{i-1} is constructed, then we may extend $s_{i-1}|_{(Z \cup U_1 \cup \dots \cup U_i) \cap U_{i+1}}$ to U_{i+1} and glue this extension with s_{i-1} to get s_i . This procedure stops and eventually gives a section s_k over X extending s .

Now suppose X is arbitrary and the $\{U_\alpha\}$ are an open covering. Let s be a section over a compact set Z , and let Z' be a compact set containing Z in its interior. Note that there is a finite compact covering of Z' on which \mathcal{F} is soft, by refining the $\{U_\alpha\}$. We can extend the section $\tilde{s} \in \Gamma(Z \cup \partial Z', \mathcal{F})$ given by s on Z and 0 on $\partial Z'$ to Z' by what has already been proved; this extends automatically by zero to all of X . □

3.3. Cohomology and filtered colimits. It is a basic fact that sheaf cohomology commutes with filtered colimits on a noetherian space, see [Har77]. Of course, the spaces of interest here (locally compact Hausdorff spaces) are anything but noetherian. Nonetheless, we shall find useful the following result:

Proposition 3.10. *The functors $H_c^i(X, \cdot)$ commute with filtered colimits of sheaves.*

Note that this already fails with $i = 0$ if ordinary cohomology is used. For instance, the global section functor does not commute with arbitrary direct sums: take for instance $X = \mathbb{R}$, and the inclusions $(i_n)_* \mathbb{Z}$ where $i_n : \{n\} \rightarrow \mathbb{R}$ is the inclusion. Then

$$\Gamma(X, \bigoplus_{\mathbb{Z}} (i_n)_* \mathbb{Z}) = \prod_{\mathbb{Z}} \mathbb{Z} \neq \bigoplus_{\mathbb{Z}} \mathbb{Z}.$$

Proof. Let us first prove this for $i = 0$. We need to show that if $\{\mathcal{F}_\alpha\}$ is a filtered system of sheaves, then

$$\varinjlim \Gamma_c(X, \mathcal{F}_\alpha) \rightarrow \Gamma_c(X, \varinjlim \mathcal{F}_\alpha)$$

is an isomorphism.

Let $t_{\beta\gamma}, \beta \leq \gamma$, be the transition maps of the directed system.

- (1) (Injectivity) Suppose $s \in \Gamma_c(X, \mathcal{F}_\alpha)$ maps to zero in the filtered colimit. We need to show that it maps to zero in some \mathcal{F}_β . By assumption (since \varinjlim commutes with stalks), for each $x \in X$ there is a neighborhood N_x of x and a β_x such that $t_{\alpha\beta_x}(s)|_{N_x} = 0$. We can find a finite collection of the $\{N_x\}$ that cover $\text{supp}(s)$, and find a β majoring them all. Then $t_{\alpha\beta}(s) = 0$.
- (2) (Surjectivity) Suppose given $\sigma \in \Gamma_c(X, \varinjlim \mathcal{F}_\alpha)$. We must show that it comes from some \mathcal{F}_β . Let Z be a compact set containing the support of σ in its interior. For each $x \in Z$ there is a neighborhood N_x of x , a β_x , and a section $s_x \in \mathcal{F}_{\beta_x}(N_x)$ mapping to σ over N_x . We can find finitely many (say x_1, \dots, x_N) that cover Z .

Moreover, choosing β larger than all the β_{x_i} , and even larger to make the s_{x_i} glue on the finitely many intersections $N_{x_i} \cap N_{x_j}$, we can obtain a section s over Z that maps to σ . Choosing β even larger, we can assume $s|_{\partial Z} = 0$, so that s extends to the entire space as a compactly supported section.

To prove it for higher dimensions, we shall use:

Lemma 3.11. *The filtered colimit of soft sheaves is soft.*

Proof. Let $\{\mathcal{F}_\alpha\}$ be a directed system of soft sheaves. We know then that for each α and $Z \subset X$ closed,

$$\Gamma_c(X, \mathcal{F}_\alpha) \rightarrow \Gamma_c(Z, \mathcal{F}_\alpha)$$

is a surjection. Taking the colimit and using the $i = 0$ case of the above result, we find that

$$\Gamma_c(X, \varinjlim \mathcal{F}_\alpha) \rightarrow \Gamma_c(Z, \varinjlim \mathcal{F}_\alpha)$$

is a surjection as well. This implies the colimit is soft. \square

The remainder of the proof is now formal, just as in [Har77]. Namely, we can consider the category of filtered systems (by a given fixed filtering category) of elements of $\mathbf{Sh}(X)$. On this we have two δ -functors given by

$$\{\mathcal{F}_\alpha\} \mapsto H_c^\bullet(X, \varinjlim \mathcal{F}_\alpha), \quad \{\mathcal{F}_\alpha\} \mapsto \varinjlim H^\bullet(X, \mathcal{F}_\alpha).$$

They are isomorphic in degree zero, and moreover both are effaceable. The first is effaceable because we can imbed any filtered system into a filtered system of soft sheaves (e.g. using the Godement

resolution), and the second is effaceable by the same effacement because the filtered colimit of soft sheaves is soft. Now general nonsense implies that the two δ -functors are naturally isomorphic. \square

Corollary 3.12. *Let \mathcal{C} be a filtered system of closed subsets of a topological space X . Let $F = \bigcap_{Z \in \mathcal{C}} F$. Then for any $\mathcal{F} \in \mathbf{Sh}(X)$, the map*

$$\varinjlim_{\mathcal{C}} H_c^i(Z, \mathcal{F}) \rightarrow H_c^i(F, \mathcal{F})$$

is an isomorphism.

Proof. For each closed subset $Z \subset X$, let $i_Z : Z \rightarrow X$ be the imbedding. We consider the sheaves $(i_Z)_*(i_Z)^*\mathcal{F}$ for $Z \in \mathcal{C}$. These form a direct system on X , whose colimit is $(i_F)_*(i_F)^*\mathcal{F}$. For whenever $Z_1 \subset Z_2$, there is a natural map

$$(i_{Z_2})_*(i_{Z_2})^*\mathcal{F} \rightarrow (i_{Z_1})_*(i_{Z_1})^*\mathcal{F}$$

obtained from the adjointness relation, for instance. The colimit of these is $(i_F)_*(i_F)^*\mathcal{F}$. If one applies H_c^\bullet to this relation, one gets the result. \square

3.4. Dimension bounds. We can now finally bound the dimensions of standard spaces. The proof will use the geometric nature of \mathbb{R} : namely, the fact that \mathbb{R} can be covered in many ways as $\mathbb{R} = F_1 \cup F_2$ where F_1, F_2 intersect in only a point and are closed (e.g. via two intervals).

We shall also need the **Mayer-Vietoris sequence** in sheaf cohomology. Let $\mathcal{F} \in \mathbf{Sh}(X)$ be a sheaf, and let $A, B \subset X$ be closed subsets. Then there is a short exact sequence of sheaves (with the “ i ’s” the inclusions)

$$0 \rightarrow (i_{A \cup B})_*(i_{A \cup B})^*\mathcal{F} \rightarrow (i_A)_*(i_A)^*\mathcal{F} \oplus (i_B)_*(i_B)^*\mathcal{F} \rightarrow (i_{A \cap B})_*(i_{A \cap B})^*\mathcal{F} \rightarrow 0.$$

Exactness can simply be checked on stalks, while the maps themselves are defined using the adjointness of i_* and i^* . There is consequently a long exact sequence in cohomology

$$\dots \rightarrow H_c^i(A \cup B, \mathcal{F}) \rightarrow H_c^i(A, \mathcal{F}) \oplus H_c^i(B, \mathcal{F}) \rightarrow H_c^i(A \cap B, \mathcal{F}) \rightarrow H^{i+1}(A \cup B, \mathcal{F}) \rightarrow \dots$$

Theorem 3.13. *If X is a locally compact space, then $\dim X \times \mathbb{R} \leq \dim X + 1$.*

Proof. This is of interest only when $\dim X < \infty$. Suppose conversely that there existed a sheaf $\mathcal{F} \in \mathbf{Sh}(X)$, and a nonzero cohomology class $\gamma \in H_c^m(X, \mathcal{F})$, where $m > \dim X + 1$.

Consider the class of all closed subsets $F \subset \mathbb{R}$. For each of these, there is a space $Y_F = X \times F \subset X \times \mathbb{R}$. We can restrict γ to each Y_F , obtaining a family of classes $\gamma_F \in H_c^m(Y_F, \mathcal{F})$. We know that $\gamma_{\mathbb{R}}$ is not zero.

The claim is that there is a *minimal* $F \subset \mathbb{R}$ such that γ_F is not zero. If this is true, then we will obtain the result easily. For then choose a splitting $F = F_1 \cup F_2$ where $F_1, F_2 \subsetneq F$ and $F_1 \cap F_2$ is a point (e.g. $F_1 = [t, \infty) \cap F, F_2 = (-\infty, t] \cap F$). Then there is a Mayer-Vietoris sequence

$$H_c^{m-1}(Y_{F_1 \cap F_2}, \mathcal{F}) \rightarrow H_c^m(Y_{F_1 \cup F_2}, \mathcal{F}) \rightarrow H_c^m(Y_{F_1}, \mathcal{F}) \oplus H_c^m(Y_{F_2}, \mathcal{F}) \rightarrow \dots$$

The first term is zero since $Y_{F_1 \cap F_2} \simeq X$ and $m - 1 > \dim X$. It follows that one of the restrictions of γ to F_1 or F_2 must be nonzero, contradicting minimality.

But now we need to see that there *is* such a minimal F . By Zorn’s lemma, we need to show that every totally ordered collection of closed subsets $F \subset \mathbb{R}$ with $\gamma_F \neq 0$ has an intersection on which γ is nonzero. But this is clear from Theorem 3.12. \square

Corollary 3.14. *Any manifold is of finite dimension.*

Proof. Indeed, a manifold is covered by open sets homeomorphic to \mathbb{R}^n . But $\dim \mathbb{R}^n \leq n$ by the above result. For $\dim \{*\} = 0$ clearly, and by induction we find $\dim \mathbb{R}^n \leq 1 + \dim \mathbb{R}^{n-1}$. \square

The dimension of \mathbb{R}^n is in fact n . This follows because $H_c^n(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}$, for instance; this can be seen using de Rham cohomology.

4. DUALITY

We can now enunciate the result we shall prove in full generality.

Theorem 4.1 (Verdier duality). *Let $f : X \rightarrow Y$ be a continuous map of locally compact spaces of finite dimension, and let k be a noetherian ring. Then $\mathbf{R}f_! : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Y, k)$ admits a right adjoint $f^!$. In fact, we have an isomorphism in $\mathbf{D}^+(k)$*

$$\mathbf{R}\mathrm{Hom}(\mathbf{R}f_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathbf{R}\mathrm{Hom}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$

when $\mathcal{F}^\bullet \in \mathbf{D}^+(X, k), \mathcal{G}^\bullet \in \mathbf{D}^+(Y, k)$.

Here $\mathbf{R}\mathrm{Hom}$ is defined as follows. Recall that given chain complexes A^\bullet, B^\bullet of sheaves, one may define a chain complex of k -modules $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$; the elements in degree n are given by the product $\prod_m \mathrm{Hom}(A^m, B^{m+n})$, and the differential sends a collection of maps $\{f_m : A^m \rightarrow B^{m+n}\}$ to $df_m + (-1)^{n+1}f_{m+1}d : A^m \rightarrow B^{m+n+1}$. Then $\mathbf{R}\mathrm{Hom}$ is the derived functor of Hom^\bullet , and lives in the derived category $\mathbf{D}^+(k)$ if $A^\bullet, B^\bullet \in \mathbf{D}^+(X, k)$. Since the cohomology in degree zero is given by $\mathrm{Hom}_{\mathbf{D}^+(X, k)}(A^\bullet, B^\bullet)$, we see that the last statement of Verdier duality implies the adjointness relation.

4.1. Representability. Ultimately, the existence of an adjoint to a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is equivalent to representability of the functor $C \mapsto \mathrm{Hom}_{\mathcal{D}}(FC, D)$ for each $D \in \mathcal{D}$. Following [GM03], we thus start by proving a simple representability lemma.

Lemma 4.2. *Let X be a space. An additive functor $F : \mathbf{Sh}(X, k) \rightarrow k\text{-mod}^{op}$ is representable if and only if it sends colimits to limits.*

The point of this result is that, while the ordinary functor $f_!$ is not (generally) a left adjoint, something very close to it is. That something very close will be in fact isomorphic in the derived category. We shall see this below.

Proof. One direction is always true (in any category). Suppose conversely that F is representable. The strategy is that $\mathbf{Sh}(X, k)$ has a lot of generators: namely, for each open set $U \subset X$, take $k_U = j_!(k)$ where $j : U \rightarrow X$ is the inclusion. We can define a sheaf $\mathcal{F} \in \mathbf{Sh}(X, k)$ via

$$\mathcal{F}(U) = F(k_U).$$

Since the k_U have canonical imbedding maps (if $U \subset U'$, there is a map $k_U \rightarrow k_{U'}$), it is clear that \mathcal{F} is a presheaf. \mathcal{F} is in fact a sheaf, though. To see this, let $\{U_\alpha\}$ be an open covering of U ; then there is an exact sequence of sheaves

$$\prod_{\alpha, \beta} k_{U_\alpha \cap U_\beta} \rightarrow \prod_{\alpha} k_{U_\alpha} \rightarrow k_U \rightarrow 0,$$

which means that there is an exact sequence

$$0 \rightarrow F(k_U) \rightarrow \prod_{\alpha} F(k_{U_\alpha}) \rightarrow \prod_{\alpha, \beta} F(k_{U_\alpha \cap U_\beta}).$$

This means that \mathcal{F} is a sheaf. \mathcal{F} is a promising candidate for a representing object, because we know that

$$\mathrm{Hom}(k_U, \mathcal{F}) \simeq \mathcal{F}(U) = F(k_U).$$

Now, we need to define a distinguished element of $F(\mathcal{F})$ and show that it is universal. More generally, we can define a natural transformation $\mathrm{Hom}(\cdot, \mathcal{F}) \rightarrow F(\cdot)$. This we can do because any $\mathcal{G} \in \mathbf{Sh}(X, k)$ is *canonically* a colimit of sheaves k_U . Namely, form the category whose objects are pairs (U, s) where $U \subset X$ is open and $s \in \mathcal{G}(U)$ and whose morphisms come from inclusions $(V, s') \rightarrow (U, s)$ where $V \subset U$ and $s' = s|_V$. For each such pair define a map $k_U \rightarrow \mathcal{G}$ by the section s .

It is easy to see that this gives a representation of \mathcal{G} *functorially* as a colimit of sheaves of the form k_U . (This is a similar observation as the well-known fact that a presheaf on any small category is canonically colimit of representable presheaves.) The natural isomorphism $\mathrm{Hom}(k_U, \mathcal{F}) \simeq F(k_U)$ now extends to a natural transformation $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \simeq F(\mathcal{G})$, which is an isomorphism. Indeed, it is an isomorphism when $\mathcal{G} = k_U$, and both functors above commute with colimits. \square

4.2. A near-adjoint to $f_!$. In an ideal world, the functor $f_!$ would already have an adjoint. That is, for each $\mathcal{G} \in \mathbf{Sh}(Y, k)$, the functor

$$\mathcal{F} \mapsto \mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_! \mathcal{F}, \mathcal{G})$$

would be representable. This is not always the case, however: $f_!$ is not (probably) an exact functor, and it need not preserve colimits. Nonetheless, a slight variant of the above functor is representable:

Proposition 4.3. *If \mathcal{M} is a soft, flat sheaf in $\mathbf{Sh}(X, k)$, then the functor $\mathcal{F} \mapsto f_!(\mathcal{F} \otimes_k \mathcal{M})$ commutes with colimits. In particular, the functor $\mathcal{F} \mapsto \mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_! \mathcal{F}, \mathcal{G})$ is representable for any $\mathcal{G} \in \mathbf{Sh}(Y, k)$.*

Proof. Indeed, we know that $f_!$ commutes with filtered colimits (because compactly supported cohomology does!) and in particular with arbitrary direct sums. As a result, it suffices to show that $\mathcal{F} \mapsto f_!(\mathcal{F} \otimes_k \mathcal{M})$ is an exact functor. If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence in $\mathbf{Sh}(X, k)$, then so is $0 \rightarrow \mathcal{F}' \otimes_k \mathcal{M} \rightarrow \mathcal{F} \otimes_k \mathcal{M} \rightarrow \mathcal{F}'' \otimes_k \mathcal{M} \rightarrow 0$ by flatness. Moreover, by Theorem 3.6 the first term is soft, so the push-forward sequence

$$0 \rightarrow f_!(\mathcal{F}' \otimes_k \mathcal{M}) \rightarrow f_!(\mathcal{F} \otimes_k \mathcal{M}) \rightarrow f_!(\mathcal{F}'' \otimes_k \mathcal{M}) \rightarrow 0$$

is exact too. The representability criterion Theorem 4.2 now completes the proof. \square

It follows that given \mathcal{M} (soft, flat) and $\mathcal{G} \in \mathbf{Sh}(Y, k)$ as above, there is a sheaf $f_{\#}(\mathcal{M}, \mathcal{G}) \in \mathbf{Sh}(X, k)$ such that

$$\mathrm{Hom}_{\mathbf{Sh}(X, k)}(\mathcal{F}, f_{\#}(\mathcal{M}, \mathcal{G})) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_!(\mathcal{F} \otimes_k \mathcal{M}), \mathcal{G}).$$

This is clearly functorial in \mathcal{G} and contravariantly in \mathcal{M} . We shall use this functor $f_{\#}$ to construct the adjoint $f^!$ in Verdier duality when \mathcal{M} is replaced by a *complex* consisting of soft, flat sheaves.

4.3. Proof of Verdier duality. We are now ready to prove Verdier duality. The strategy will be to choose a soft, flat, and bounded resolution \mathcal{L}^\bullet of the constant sheaf k , so a quasi-isomorphism $k \rightarrow \mathcal{L}^\bullet$. We have already seen that we can do most of this; however, we should check that we can choose \mathcal{L}^\bullet to be *bounded*. To do this, we truncate \mathcal{L}^\bullet after the n th stage, where $n = \dim X$; the resulting complex will remain soft by Theorem 3.4. Namely, we consider the complex

$$\tau_{\leq n+1} \mathcal{L}^\bullet : 0 \rightarrow \mathcal{L}^0 \rightarrow \mathcal{L}^1 \rightarrow \dots \rightarrow \mathcal{L}^n \rightarrow \mathrm{im}(\mathcal{L}^n \rightarrow \mathcal{L}^{n+1}) \rightarrow 0.$$

Since $\mathcal{L}^1, \dots, \mathcal{L}^n$ are soft, so is the final term as $n = \dim X$. The final term is also flat because of the stalkwise split nature of the resolution \mathcal{L}^\bullet (at least if it was constructed using a Godement resolution).

Then \mathcal{F}^\bullet and $\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet$ will be isomorphic functors on the level of the derived categories, but the latter will be much better behaved; for instance, it will have soft terms. In other words, we are going to obtain a *functorial* “soft (and thus $f_!$ -acyclic) replacement” for a given object in $\mathbf{D}^+(X, k)$.

Fix a complex $\mathcal{G}^\bullet \in \mathbf{D}^+(Y, k)$. We need to show that the functor $\mathcal{F}^\bullet \mapsto \mathrm{Hom}_{\mathbf{D}^+(Y, k)}(\mathbf{R}f_!(\mathcal{F}^\bullet), \mathcal{G}^\bullet)$ is representable. However, there is a canonical isomorphism (in the derived category, or a quasi-isomorphism on the level of complexes)

$$\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet.$$

This works in the derived category (we do not need to take the derived tensor product) as \mathcal{L}^\bullet is flat and bounded. So, alternatively, we may show that the functor $\mathcal{F}^\bullet \mapsto \mathrm{Hom}_{\mathbf{D}^+(Y, k)}(\mathbf{R}f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet)$ is representable.

We shall in fact show that there is a complex $\mathcal{K}^\bullet \in \mathbf{D}^+(X, k)$ such that there is a functorial isomorphism

$$(3) \quad \mathbf{R}\mathrm{Hom}(\mathbf{R}f_!(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq \mathbf{R}\mathrm{Hom}(\mathbf{R}f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq \mathbf{R}\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{K}^\bullet).$$

This is what we want for the stronger form of Verdier duality anyway. If we wade through this notation, we notice one thing: $\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet$ is already $f_!$ -acyclic; in particular,

$$\mathbf{R}f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet) \simeq f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet)$$

where $f_!$ is applied pointwise. Moreover, we can assume that \mathcal{G}^\bullet is a complex of injectives, and certainly we can (and will) try to choose \mathcal{K}^\bullet to consist of injectives. In this case, we are just looking for a quasi-isomorphism

$$\mathrm{Hom}^\bullet(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \stackrel{qis}{\simeq} \mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

valid for any complex \mathcal{F}^\bullet . However, we know that

$$\begin{aligned} \mathrm{Hom}^n(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) &= \prod_m \prod_{i+j=m} \mathrm{Hom}(f_!(\mathcal{F}^i \otimes \mathcal{L}^j), \mathcal{G}^{m+n}) \\ &\simeq \prod_m \prod_{i+j=m} \mathrm{Hom}(\mathcal{F}^i, f_\#(\mathcal{L}^j, \mathcal{G}^{m+n})) \\ &= \prod_{i,j} \mathrm{Hom}(\mathcal{F}^i, f_\#(\mathcal{L}^j, \mathcal{G}^{i+j+n})). \end{aligned}$$

If we consider the double complex given by $C^{rs} = f_\#(\mathcal{L}^{-r}, \mathcal{G}^s)$ (with the boundary maps being those induced by \mathcal{L}, \mathcal{G} ; remember that $f_\#$ is contravariant in the first variable) and let \mathcal{K}^\bullet be the associated chain complex with $\mathcal{K}^t = \bigoplus_{r+s=t} f_\#(\mathcal{L}^{-r}, \mathcal{G}^s)$, then it follows that there is an isomorphism

$$\mathrm{Hom}^n(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq \mathrm{Hom}^n(\mathcal{F}^\bullet, \mathcal{K}^\bullet).$$

In fact, there is an isomorphism of *complexes*

$$\mathrm{Hom}^\bullet(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq \mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{K}^\bullet).$$

This follows from checking through the signs of the differential. This will prove (3) if we check that \mathcal{K}^\bullet is a bounded-below complex of injectives. It is bounded below from the definition (as \mathcal{L}^\bullet is bounded in both directions).

To see that it is injective, we recall that we had chosen \mathcal{G}^\bullet to be a complex of injectives, make the observation:

Lemma 4.4. *$f_\#(\mathcal{M}, \mathcal{G})$ is injective whenever $\mathcal{M} \in \mathbf{Sh}(X, k)$ is a soft, flat sheaf and $\mathcal{G} \in \mathbf{Sh}(Y, k)$ is injective.*

Proof. Recall that $f_{\#}(\mathcal{M}, \mathcal{G})$ is the object representing the functor

$$\mathcal{F} \mapsto \mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_!(\mathcal{F} \otimes \mathcal{M}), \mathcal{G}).$$

To say that it is injective is to say that mapping into it is an exact functor, or simply that it is a *right exact* functor. Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F}$ be an exact sequence. Then

$$0 \rightarrow f_!(\mathcal{F}' \otimes \mathcal{M}) \rightarrow f_!(\mathcal{F} \otimes \mathcal{M})$$

is exact too, so injectivity of \mathcal{G} gives that

$$\mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_!(\mathcal{F} \otimes \mathcal{M}), \mathcal{G}) \rightarrow \mathrm{Hom}_{\mathbf{Sh}(Y, k)}(f_!(\mathcal{F}' \otimes \mathcal{M}), \mathcal{G}) \rightarrow 0$$

is also exact. This statement is the meaning of injectivity. \square

It is now clear how we may define the functor $f^! : \mathbf{D}^+(Y, k) \rightarrow \mathbf{D}^+(X, k)$. Given a bounded-below complex $\mathcal{G}^\bullet \in \mathbf{D}^+(Y, k)$, we start by replacing it with a complex of injectives, and so just assume that it consists of injectives without loss of generality. We then form the complex \mathcal{K}^\bullet of sheaves on X such that $\mathcal{K}^t = \bigoplus_{r+s=t} f_{\#}(\mathcal{L}^{-r}, \mathcal{G}^s)$, where \mathcal{L}^\bullet is a fixed soft resolution of the constant sheaf. Then setting $f^! \mathcal{G}^\bullet = \mathcal{K}^\bullet$ finishes the proof, by (3); we have functoriality built in.

5. DUALITY ON MANIFOLDS

In this section, we shall apply the existence of $f^!$ to questions involving manifolds. Once we know that $f^!$ exists, we will be able to describe it using the adjoint property rather simply (for manifolds). This will lead to clean statements of theorems in algebraic topology. For instance, Poincaré duality will be a direct consequence of the fact that, on an n -dimensional oriented manifold, the dualizing sheaf (see below) is just $k[n]$.

5.1. The dualizing complex. After wading through the details of the proof of Verdier duality, let us now consider the simpler case where $Y = \{*\}$. X is still a locally compact space of finite dimension, and k remains a noetherian ring. Then Verdier duality gives a right adjoint $f^!$ to the functor $\mathbf{R}\Gamma_c : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(k)$. In other words, for each $\mathcal{F}^\bullet \in \mathbf{D}^+(X, k)$ and each complex G^\bullet of k -modules, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma_c(\mathcal{F}^\bullet), G^\bullet) \simeq \mathrm{Hom}_{\mathbf{D}^+(X, k)}(\mathcal{F}^\bullet, f^!(G^\bullet)).$$

Of course, the category $\mathbf{D}^+(k)$ is likely to be much simpler than $\mathbf{D}^+(X, k)$, especially if, say, k is a field.

Definition 5.1. $\mathcal{D}^\bullet = f^!(k)$ is called the **dualizing complex** on the space X . \mathcal{D}^\bullet is an element of the derived category $\mathbf{D}^+(X, k)$, and is well-defined there. We will always assume that \mathcal{D}^\bullet is a bounded-below complex of *injective* sheaves.

In fact, \mathcal{D}^\bullet can always be taken to be a *bounded* complex of injective sheaves, though we shall not need this.

We now want to determine the properties of this dualizing complex, and show in particular that we can recover Poincaré duality when X is a manifold. To do this, let us try to compute the cohomology $H^\bullet(\mathcal{D}^\bullet)$ of the dualizing complex (which will be a collection of sheaves). The i th cohomology can be obtained as the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathbf{D}^+(X, k)}(k_U, \mathcal{D}^\bullet[i]).$$

Here, as usual $k_U = j_!(k)$ is the extension by zero of the constant sheaf k from U to X .

Indeed, to check this relation we recall that we assumed \mathcal{D}^\bullet a complex of injectives (without loss of generality), so that maps $k_U \rightarrow \mathcal{D}^\bullet[i]$ are just homotopy classes of maps $k_U \rightarrow \mathcal{D}^\bullet[i]$, or equivalently (by the universal property of k_U) elements of $H^0(\mathcal{D}(U)^\bullet[i])$. But the sheaf associated to this presheaf is clearly the homology $H^i(\mathcal{D}^\bullet)$.

We have proved in fact:

Proposition 5.2. *If $\mathcal{F}^\bullet \in \mathbf{D}^+(X, k)$, the cohomology $H^i(\mathcal{F}^\bullet) \in \mathbf{Sh}(X, k)$ is the sheaf associated to the presheaf $\mathrm{Hom}_{\mathbf{D}^+(X, k)}(k_U, \mathcal{F}^\bullet[i])$.*

So we need to compute $\mathrm{Hom}_{\mathbf{D}^+(X, k)}(k_U, \mathcal{D}^\bullet[i]) = \mathrm{Hom}_{\mathbf{D}^+(X, k)}(k_U[-i], \mathcal{D}^\bullet)$. By taking U small, we may assume that U is a ball in \mathbb{R}^n . From the adjoint property, however, this is feasible: such maps are in natural bijection with maps $\mathbf{R}\Gamma_c(k_U[-i]) \rightarrow k$ in $\mathbf{D}^+(k)$. So we need to compute $\mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma_c(k_U[-i]), k)$. Here $\mathbf{R}\Gamma_c(k_U[-i])$ is represented by a bounded complex.

One might hope that this is somehow related to the compactly supported cohomology of U . When k is a field, every complex is quasi-isomorphic to its cohomology, and it is true. We get:

Proposition 5.3. *If k is a field, then the i th cohomology of the dualizing complex $H^i(\mathcal{D}^\bullet)$ is the sheaf associated to the presheaf $U \mapsto H_c^i(U, k)^\vee$.*

Chasing through the definitions, one sees that the restriction maps are the duals to the maps $H_c^i(U, k) \rightarrow H_c^i(U', k)$ for $U \subset U'$. (This goes the opposite way as in ordinary cohomology.)

We in particular see that \mathcal{D}^\bullet lives in the *bounded* derived category, at least when k is a field (because \mathcal{D}^\bullet is now quasi-isomorphic to a suitable truncation).

The next goal will be to compute the cohomology of \mathcal{D}^\bullet for a manifold. For a field at least, this will require nothing more than a computation of the cohomology of suitable open sets in \mathbb{R}^n , by the previous result. For more general rings, we will have to invoke the hypercohomology spectral sequence. This will require a significant digression.

5.2. The cohomology of \mathbb{R}^n . The point of Poincaré duality lies in the cohomology of \mathbb{R}^n and the fact that any manifold is locally homeomorphic to \mathbb{R}^n . Of course, we mean *compactly* supported cohomology here.

Lemma 5.4. *Let k be any ring. Then we have $H_c^i(\mathbb{R}^n, k) \simeq k$ if $i = n$, and $H_c^i(\mathbb{R}^n, k) = 0$ otherwise.*

Proof. We refer the reader to [Ive86]. The strategy, in rough outline, is as follows:

- (1) It is sufficient to handle the case $k = \mathbb{Z}$, because then it is clear for any free abelian group, and one can use the exact sequences to deduce it in general (together with the fact that cohomology commutes with filtered colimits).
- (2) One shows that $H_c^\bullet([0, 1], \mathbb{R}) = H^\bullet([0, 1], \mathbb{R})$ is \mathbb{R} in dimension zero and zero otherwise. This follows, for instance, by use of the soft *de Rham resolution*

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty \rightarrow 0,$$

where \mathcal{C}^∞ denotes the sheaf of smooth functions and the last map is one-variable differentiation. In particular, $H_c^1([0, 1], \mathbb{R})$ can be computed as the cokernel of differentiation

$$\mathcal{C}^\infty([0, 1]) \xrightarrow{f \mapsto f'} \mathcal{C}^\infty([0, 1]),$$

which is clearly trivial.

- (3) One computes $H_c^\bullet([0, 1], \mathbb{Z})$ using the soft(!) resolution

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{C} \rightarrow \mathcal{C}^{S^1} \rightarrow 0,$$

where \mathcal{C} is the sheaf of real-valued continuous functions and \mathcal{C}^{S^1} is the sheaf of continuous functions into the circle group. One can deduce from this (and the so-called Vietoris-Bergle mapping theorem) that sheaf cohomology is a homotopy invariant.

- (4) In the end, one can show that sheaf cohomology with coefficients in the constant sheaf \mathbb{Z} is a *cohomology theory* (satisfying, that is, the usual Eilenberg-Steenrod axioms) on suitably nice spaces. The analog of relative cohomology is local cohomology. Because of the normalization of the cohomology of a point, it follows that $H^\bullet(S^n, \mathbb{Z})$ is \mathbb{Z} in dimensions 0 and n , and zero otherwise.
- (5) Finally, to compute $H_c^\bullet(\mathbb{R}^n, \mathbb{Z})$, one uses the fact that the one-point compactification of \mathbb{R}^n is S^n and the long exact sequence.

□

It follows that the same is true when \mathbb{R}^n is replaced by any space homeomorphic to it, e.g. an open ball. Using this, we make the following observation: if X is an n -dimensional manifold, then the sheaf associated to the presheaf $U \mapsto H_c^i(U, k)^\vee$ is zero unless $i = n$. It follows that the dualizing complex \mathcal{D}_X^\bullet is cohomologically concentrated in *one* degree, namely n . It follows (by the use of truncation functors) that the dualizing complex is quasi-isomorphic to a translate of a single sheaf.

On a n -dimensional manifold X , we define the **orientation sheaf** ω_X as the sheaf associated to the presheaf $U \mapsto H_c^n(U, k)^\vee$. (This is actually already a sheaf, though we do not need this.)

Corollary 5.5. *Let k be a noetherian ring, and let X be an n -dimensional manifold. Then the dualizing complex \mathcal{D}^\bullet on X is isomorphic to $\omega_X[n]$.*

Proof. We have to be careful here. In theorem 5.3, we computed the cohomology of the dualizing complex with coefficients in a *field*. The only place we used that k was a field there was in the last step. In general, we have that $H^i(\mathcal{D}^\bullet)$ is the sheaf associated to the presheaf $\mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma_c(k_U[-i]), k)$. Since in any case, for U a ball, $\mathbf{R}\Gamma_c(k_U[-i])$ is concentrated in one degree (as a *projective* k -module), we find that $H^i(\mathcal{D}^\bullet)$ is in fact the sheaf associated to the presheaf $H_c^i(U, k)^\vee$. In other words, we have proved theorem 5.3 with coefficients in *any* noetherian ring. □

5.3. Poincaré duality. Fix a *field* k . Let X be an n -dimensional manifold with orientation sheaf ω_X . We know that the dualizing sheaf is $\omega_X[n]$, which implies for any complex $\mathcal{F}^\bullet \in \mathbf{D}^+(X, k)$, there is a natural isomorphism $\mathrm{Hom}_{\mathbf{D}^+(X, k)}(\mathcal{F}^\bullet, \omega_X[n]) \simeq \mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma_c(\mathcal{F}^\bullet), k)$.

Take in particular $\mathcal{F}^\bullet = \mathcal{H}[r]$ for some $r \in \mathbb{Z}$ and $\mathcal{H} \in \mathbf{Sh}(X, k)$. On the left, we get $\mathrm{Ext}_k^{n-r}(\mathcal{H}, \omega_X)$; on the right, we get (since k is a field) $H_c^r(X, \mathcal{H})^\vee$.

Theorem 5.6 (Poincaré duality). *There is a map*

$$\int_c : H_c^n(X, \omega_X) \rightarrow k$$

such that, for any k -sheaf \mathcal{H} , the pairing

$$H_c^r(X, \mathcal{H}) \times \mathrm{Ext}_k^{n-r}(\mathcal{H}, \omega_X) \mapsto H_c^n(X, \omega_X) \xrightarrow{\int_c} k$$

identifies $H_c^r(X, \mathcal{H})^\vee \simeq \mathrm{Ext}_k^{n-r}(\mathcal{H}, \omega_X)$.

When \mathcal{H} is the *constant* sheaf k , then $\mathrm{Ext}_k^{n-r}(\mathcal{H}, \omega_X) = H^{n-r}(X, \omega_X)$ (this is *not* compactly supported cohomology!) and consequently one finds that there is a natural isomorphism $H_c^r(X, k)^\vee \simeq H^{n-r}(X, \omega_X)$. When X is orientable and ω_X is the constant sheaf k , then we have recovered the usual form of Poincaré duality.

Corollary 5.7. *Let X be an n -dimensional manifold. The functor $\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$ on $\mathbf{Sh}(X, k)$ is representable.*

Proof. Indeed, the representing object is ω_X , as follows from Poincaré duality above. \square

5.4. Relative cohomology. Let X be a topological space, and $Z \subset X$ a closed subspace. Sheaf cohomology of the constant sheaf k on X is to be thought of as an analog to the singular cohomology $H_{\text{sing}}^\bullet(X; k)$; in fact, these coincide for a nice space X . The analog of the *relative* singular cohomology $H_{\text{sing}}^\bullet(X, Z; k)$ are the *local cohomology* groups $H_Z^\bullet(X, k)$ for k the constant sheaf.

Namely, consider the functor that sends $\mathcal{F} \in \mathbf{Sh}(X)$ to $\Gamma_Z(\mathcal{F})$, the group of global sections with support in Z . The derived functors $H_Z^i(X, \mathcal{F})$ are called the **local cohomology groups** of \mathcal{F} .

We recall that if $i : Z \rightarrow X$ is the inclusion, then we showed much earlier (Theorem 1.1) that the push-forward i_* had a right adjoint $i^!$. Since (as is easy to see),

$$\Gamma_Z(\mathcal{F}) = \text{Hom}_{\mathbf{Sh}(X)}(i_*\mathbb{Z}, \mathcal{F}),$$

we get by adjointness

$$\Gamma_Z(\mathcal{F}) = \text{Hom}_{\mathbf{Sh}(Z)}(\mathbb{Z}, i^!\mathcal{F}).$$

In other words, the global sections of $i^!\mathcal{F}$ are precisely $\Gamma_Z(\mathcal{F})$; this is also immediate from the actual construction of $i^!$ we gave.

Our notation is, however, slightly confusing! We have defined $i^! : \mathbf{Sh}(X, k) \rightarrow \mathbf{Sh}(Z, k)$ as the right adjoint to i_* . However, we also used the notation $i^!$ for the right adjoint to the *derived* functor $\mathbf{R}i_! = \mathbf{R}i_* : \mathbf{D}^+(Z, k) \rightarrow \mathbf{D}^+(X, k)$. The next lemma will show that our abuse of notation is not as bad as it may seem.

Lemma 5.8. *Let $i^! : \mathbf{Sh}(X, k) \rightarrow \mathbf{Sh}(Z, k)$ be right adjoint to i_* . Then $\mathbf{R}i^! : \mathbf{D}^+(X, k) \rightarrow \mathbf{D}^+(Z, k)$ is right adjoint to $\mathbf{R}i_*$ (so it is the “upper shriek” of before).*

Proof. Now $i^! : \mathbf{Sh}(X, k) \rightarrow \mathbf{Sh}(Z, k)$ is a left-exact functor (as a right adjoint). Since its left adjoint i_* is *exact*, a simple formal argument shows that $i^!$ preserves injectives. From this the argument is straightforward. Namely, let $\mathcal{F}^\bullet \in \mathbf{D}^+(X, k)$, $\mathcal{G}^\bullet \in \mathbf{D}^+(Z, k)$. We may assume that these are bounded-below complexes of injective sheaves. Then we have that $\mathbf{R}i_*(\mathcal{F}^\bullet) = i_*(\mathcal{F}^\bullet)$ and $\mathbf{R}i^!(\mathcal{G}^\bullet) = i^!(\mathcal{G}^\bullet)$. We have that

$$\text{Hom}_{\mathbf{D}^+(Y, k)}(\mathbf{R}i_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet) = \text{Hom}_{\mathbf{D}^+(Y, k)}(i_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet) = [i_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet],$$

where the last symbol denotes homotopy classes; this follows because \mathcal{G}^\bullet is injective. Similarly we get

$$\text{Hom}_{\mathbf{D}^+(X, k)}(\mathcal{F}^\bullet, \mathbf{R}i^!(\mathcal{G}^\bullet)) = \text{Hom}_{\mathbf{D}^+(X, k)}(\mathcal{F}^\bullet, i^!(\mathcal{G}^\bullet)) = [\mathcal{F}^\bullet, i^!(\mathcal{G}^\bullet)]$$

because $i^!$ preserves injectives. Now by adjointness on the ordinary categories, we see the natural isomorphism. \square

Now we want to bring in Verdier duality.

Lemma 5.9. *Let X, Y, Z be locally compact spaces of finite dimension, and suppose $f : X \rightarrow Y, g : Y \rightarrow Z$ are continuous maps. Then $(g \circ f)^! = f^! \circ g^! : \mathbf{D}^+(Z, k) \rightarrow \mathbf{D}^+(X, k)$.*

To be precise, we should say “up to a natural isomorphism.”

Proof. This is immediate from the adjointness definition of the upper shriek and the fact that

$$\mathbf{R}(g \circ f)_! = \mathbf{R}g_! \circ \mathbf{R}f_!$$

(which is the Leray spectral sequence Theorem 2.15). \square

We can deduce the following. If \mathcal{D}_X^\bullet is the dualizing complex on X , then $\mathbf{R}i^!(\mathcal{D}_X^\bullet)$ is the dualizing complex \mathcal{D}_Z^\bullet on Z . This follows because the dualizing complex is the upper shriek of the constant complex k . Suppose now X is a manifold of dimension n , with orientation sheaf ω_X . Then it follows that

$$\mathcal{D}_Z^\bullet = \mathbf{R}i^!(\omega_X)[n].$$

From this we will get:

Theorem 5.10 (Alexander duality). *Let k be a field. Suppose $Z \subset X$ is a closed subset of an n -dimensional manifold X with $i : Z \rightarrow X$ the inclusion. Then, for a sheaf $\mathcal{F} \in \mathbf{Sh}(Z, k)$, we have natural isomorphisms:*

$$H_c^r(Z, \mathcal{F})^\vee \simeq \mathrm{Ext}_k^{n-r}(i_*\mathcal{F}, \omega_X[n-r]).$$

The most important case is when $\mathcal{F} = k$, in which case we find:

$$H_c^r(Z, k)^\vee \simeq H_Z^{n-r}(X, \omega_X).$$

Proof. Indeed, this follows purely formally now. Let $\mathcal{F} \in \mathbf{Sh}(Z)$ be any sheaf. In the following, we shall interchange i_* and $\mathbf{R}i_*$, which is no matter as i_* is an exact functor. Then:

$$\begin{aligned} H_c^r(Z, \mathcal{F})^\vee &= \mathrm{Hom}_{\mathbf{D}^+(k)}(\mathbf{R}\Gamma_c(\mathcal{F}), k[-r]) \\ &= \mathrm{Hom}_{\mathbf{D}^+(Z, k)}(\mathcal{F}, \mathcal{D}_Z[-r]) \\ &= \mathrm{Hom}_{\mathbf{D}^+(Z, k)}(\mathcal{F}, \mathbf{R}i^!\mathcal{D}_X[-r]) \\ &= \mathrm{Hom}_{\mathbf{D}^+(Z, k)}(\mathcal{F}, \mathbf{R}i^!\omega_X[n-r]) \\ &= \mathrm{Hom}_{\mathbf{D}^+(X, k)}(i_*\mathcal{F}, \omega_X[n-r]) \end{aligned}$$

Take now $\mathcal{F} = k$. We then get at the last term $\mathrm{Ext}_k^{n-r}(i_*k, \omega_X)$. But recall that these are the local cohomology groups $H_Z^{n-r}(X, \omega_X)$ because $\mathrm{Hom}_{\mathbf{Sh}(X, k)}(i_*k, \cdot)$ is the same functor as $\Gamma_Z(\cdot)$. \square

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