Bubbling, Compactness, Nonsqueezing

Thm: For \( l \geq 2, p > 2, \) \( J \in \mathcal{C}^l(\Sigma), \)
\( u : \Sigma \to M \) an \( L^p, J\)-hol. curve.
Then \( u \) is in fact \( L^\infty \) (we smooth \( f \to \infty \))
("elliptic regularity")

Thm: For \( l \geq 2, p > 2, \) \( J_k \in \mathcal{C}^l(\Sigma) \to J \in \mathcal{C}^l(\Sigma), \)
\( (\Sigma, J_k) \to (\Sigma, J). \) \( u_k : \Sigma \to M \) \( J_k\)-hol. bd
\( \epsilon \)

[MS, 173] Then on compact subsets of \( \Sigma, J \) subseq. converg. in \( \mathcal{C}^{l-1}. \)

Now: Consider \( u_k \) \( J\)-hol., in homology class \( A, \)
then bounded in \( L^2. \)
Since smooth by above, can only fail to
converge in \( L^p; \) if \( \sup_{z \in Z} |dv_k(z)| \to \infty. \)

Let \( z^k \) a point at which \( |dv_k| \) maximized,
May assume \( z_k \to z_0. \)

Choose a hol. chart \( \varphi: \mathcal{N} \to \Sigma \)
\( 0 \to z_0. \)

\( v^k(z) = \varphi \circ u^k (z + z_k/c_k) \)
\( c_k \) chosen so \( |dv^k(0)| = 1. \)

\( \implies \) bad energy seq on growing subset
of \( C. \) By remov. seq.,
yet \( v^\infty : C' \to M \)
Thus we have:

**Prop:** If \( A \in H^1(M) \), \( w(A) > 0 \) is such that \( A \perp A' \) with \( 0 < w(A') < w(A) \), then \( M(A, \epsilon, J) \) compact.

**Full Thm:** For \( k \geq 2 \), \( p > 2 \), \( J_k \in C^0(\Sigma) \) \((\text{Gromov Compactness})\), \( u_k : \Sigma \rightarrow M \), \( J_k \) J-holo \((\text{with } [u^*_k \Sigma] = A \text{ fixed})\). Suppose \( u_k \) s.t. \( \lim \nabla_{J_k} u_k \) (collapse inessential loops to points)

\[ \lim E(u_k) = \Sigma E(\text{pts}) \]

(one more technical point: "bubbles connect")

**Link:** if don't require \( J_k \rightarrow J \), can collapse essential loops too.

We omit the proof of this "full thm." we mostly have use for the proposition.
Proof:

2: get \( B^{2n}(r) \to S^{2}(\pi \{ \pi(\mathbb{R}^2) \}) \times R^{2n-2} / K \Pi^{2n-2} \),

then \( r \leq R \).

\# J-hol. spheres in \( S^2 \times T^{2n-2} \) in class \((S^2, o)\)

through a point \((x, y) \in S^2 \times T^{2n-2} \) is 1

for \( J_{st} = J_0 \times J_T \), standard

(will show\( \) generic\( \) below)

Thus \( \equiv 1 \) (mod 2), for \( J \) given by \( J_0 \) from \( S^{2n} \) on \( i(B^{2n}(r - \varepsilon)) \), extended \( \equiv 0 \) \( \) (show\( \) generic\( \) below)

By\( \) monotonicity, \( \text{area} \geq \pi \( (r - \varepsilon)^2 \)

\( \pi (r + \varepsilon)^2 \to \varepsilon \to 0 \) \( r \leq R \).

(Recall\( \) monotonicity for \( R^k \):

In \( R^k \), if \( 0 \in S \), then

\( \text{Area}(S \cap B_0(r)) \geq \pi r^2 \)

Genericity for \( J \): Any hole sphere exits \( i(B^{2n}(r)) \).

Somewhere\( \prime \)jective points are dense, so \( J \) some outside of \( i(B^{2n}(r)) \).
Consider space \( J_i \) of a.c.i.s. \( J \) agreeing with \( J \) from \( \mathfrak{B}^m(\mathbb{R}) \) on \( \mathfrak{B}^m(\mathbb{R} - \mathbb{S}) \).

**Claim:** \( \forall \omega, (D, d\omega) : T J_i \times L^1(\Sigma, \omega, TM) \rightarrow L^0(\Sigma, \Sigma, \omega, TM) \times TM \)

is surjective.

**Proof:** Consider element \( \gamma \). \( D^x \gamma = 0 \) so \( \gamma \) smooth, soln to a real Cauchy-Riemann eqn.

Because somewhere inj points are dense in \( \mathfrak{m}(\Sigma) \cap (\mathfrak{B}^m(\mathbb{R} - \mathbb{S})) \), a nonempty open subset of \( \mathfrak{m}(\Sigma) \), we have that \( \gamma = 0 \) there. Thus by unique cont for solns to real CR eqns, \( \gamma = 0 \) everywhere.

Thus \( \exists \) generic choice \( \tilde{J} \).

Check index:

index of halo spheres \( \in \text{class } (S^2, 0) \):

\[
\mathfrak{r}(S^2) + 2 \int \left( S^2 \times T^{m-2} \right) \cdot [S^2] = 2n + 4
\]

Index of \( (D, d\omega) \) is \( \text{real } (\text{dim } TM) = 2n + 4 - 2m = 4 \)

Now need to look at \( \text{Aut } \left( (\Sigma, x) \right), \) (aut of \( \Sigma = S^2 \) fixing)

4-dimensional.

Yet: “expected dimension” of halo spheres thru a pt is \( 0 \), no isolated pts.
This is what we see for $L$, let us check:

**Genericity for $L$:** We'll show the $2n+4$-dim space is cut out smoothly, then observe transversality for $L$.

We have a splitting as $\Sigma \oplus (\Sigma)^* \oplus K$, for sum of line bundles.

Furthermore, $D$ respects the splitting

$$D: \oplus L_i^2 (\Sigma, L_i) \to \oplus L_i^2 (\Sigma, L_i^0, \Sigma \oplus L_i)$$

$$\ker D = \oplus H^0 \left( \Sigma, L_i \right) \quad \text{coker } D = \oplus H^1 \left( \Sigma, L_i \right)$$

**Some duality:**

$$H^1 (\Sigma, L_i^*) = H^0 \left( \Sigma, L_i \right) \otimes K$$

where $H_2^{n+1} (\Sigma, L_i^* \otimes K) \cong H^0 \left( \Sigma, L_i^* \right) \cong H^0 (\Sigma, L_i \otimes K)$

$$H^0 (\Sigma, L_i) \text{ nonzero } \Rightarrow c_1 (L) \geq 0$$

$$c_1 (L^* \otimes K) = -c_1 (L) - c_1 (K) = -c_1 (L) - 2$$

Thus $c_1 (L) \text{ nonzero } \Rightarrow c_1 (L) \leq -2$

Thus $\text{coker } D = 0$ since $c_1 (L_0) = 2, c_i (L_i) = 0, i \neq 0$.

**Rank:** "automatic transversality in dim 4".

**Prop:** Holomorphic sphere $m: \mathbb{P}^r \to M$, somewhere $m$ injective, $D_m$ onto

$$\Leftrightarrow c_1 (m^* TM) \geq 1$$

**Idea:** $m^* TM = T\mathbb{P}^r \oplus N, c_1 (m^* TM) \geq 1 \Leftrightarrow c_1 (N) \geq -1$

$$\Rightarrow D_m \text{ onto.}$$

(need for $c_1$ also too, but OK)
Consider $(S^2 \times S^2, \omega \oplus \omega)$.

**Theorem 1 (Gromov):** $\text{Symp}(S^2 \times S^2, \omega \oplus \omega)$ has two components. The $\alpha$-cpt is homotopy equivalent to $SO(3) \times SO(3)$.

**Theorem 2 (Gromov):** $\text{Symp}(\mathbb{C}P^2, \omega_{FS})$ is homotopy equivalent to $\text{PU}(2)$.

**Proof of 1:** $\text{Symp}_0 \times J \to J$

\[ \psi : J \to J \psi^{-1} \psi \psi^{-1} = \psi \psi^{-1} \]

**Remark:** $\text{Symp}_0 \times \mathbb{Z}/3 \to J$ infinite codimension (e.g. $\text{Int} \psi^{-1} \text{Int} \psi$)

$\text{Symp}_0 \times \mathbb{Z}/3 \to J$ stabilizer is $SO(3) \times SO(3)$

$\text{Symp}_0 \to J$ (not surj). We provide a left inverse

Then centre of $J = \text{Centre} \text{Symp}_0 \times SO(3)$.

Held curves in class $A = [S^1 \times \mathbb{R}^2]$, $B = [S^1 \times S^2]$: automated tautness $A \cong J$. Thus we get smooth

$2\chi(S^2) + 2c_1(S^2 \times S^2) \cdot A = 4 + 4 = 8$

($\text{mod } \text{Aut}(\mathbb{C}P^1)$ is only $2$-twist)

Together with evaluation map, have $8 - 4 = 4$

but then mod $\text{Aut}(\mathbb{C}P^1)$ gives $0$.

Count mod 2 is 1 so $3$ hole sphere than every pt.

Can't have 0 effective transverse, but not 1. If have 1 can't have more (by proposition of int. as below.)
Claim: We get 2 filtrations by spheres $F_A$, $F_B$ with a leaf of $F_A$ not a leaf of $F_B$ in one point.

Proof: Positivity of intersections (works even in non-integrable case --- see [W2, 4.2.3])

Thus 2 spheres in class $A$ coincide or are disjoint,
similarly for $B$ (since $A.A = B.B = 0$)
and since $A.B = 1$, only 1 N pt.

Choose params of the two curves thru (0,0).

We have $\mathbb{R} \times 1^2 \text{param} \rightarrow \text{Sym}_2 \rightarrow \text{SO}(3) \text{ Sym}_2 \rightarrow \text{SO}(3) \times \text{SO}(3)$ using equiv.

Given from filtrations of params.