Holomorphic Curves

Let \((M^{2n}, \omega, J)\) be a \(2n\)-dimensional compact symplectic manifold with compact \(a.c.s. J\).

Let \((\Sigma, j)\) be a Riemann surface.

We say \(u: \Sigma \to M\) is \(J\)-holomorphic if \(j \circ du = du \circ j\).

i.e. \(j \circ du \circ j = -du\), i.e.

\[\overline{\partial_J} u := \frac{1}{2} (du + j \circ du \circ j) = 0\]

\(\overline{\partial_J} u\) is a section of \(\text{End} (T\Sigma, u^* TM)\), i.e.

\[\mathcal{L} (\Sigma, u^* TM) = \mathcal{L}_J^{1, 0} (\Sigma, u^* TM) \oplus \mathcal{L}_J^{0, 1} (\Sigma, u^* TM)\]

\(\alpha\)-linear \(\psi\)-anti-linear

\[\overline{\partial_J} u = \pi_0^{1, 0} (du)\] \(\Leftrightarrow\) \(\overline{\partial_J} u = 0 \Leftrightarrow du = \mathcal{L}_J^{1, 0} (\Sigma, u^* TM)\)

In coordinates \((s, t)\) on \(\Sigma\) for which \(\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\):

\[0 = \overline{\partial_J} u (\frac{\partial}{\partial s}) = \frac{1}{2} \left( \frac{\partial u}{\partial s} + J(u(s, t)) \frac{\partial u}{\partial t} \right)\]

\[0 = \overline{\partial_J} u (\frac{\partial}{\partial t}) = \frac{1}{2} \left( \frac{\partial u}{\partial t} - J(u(s, t)) \frac{\partial u}{\partial s} \right)\] (same)
Goal: Given $A \in H_2(M; \mathbb{Z})$, study $M(A, \Sigma, J) = \{ u \in C^0(\Sigma, M) \mid J_u = du \}$. Also, given $A \in H_2(M; \mathbb{L}; \mathbb{Z})$; study $\\{ u^\ast \Sigma \} = A$.

We'll want conditions under which this is:

1. cut-out transversally for generic $J$
2. is compact (or, failing that, understand its failure to be)
3. and will want to know its dimension.

(b) giving
(c) allow non-auto

deep this for Morse flow lines:

\[ \delta_t - \Delta(t) : L^2 \rightarrow L^2 \text{ for } \delta \text{ regular of index} \]

plus (a) surjectivity of linearization given by $\delta_t$ of map $\Sigma \times L^2 \rightarrow L^2$.

(b) We pulled a trick here. For more generalizable arg would be:

(61) $L^2 \rightarrow C^0$ and $L^2 \rightarrow C^{k-1}$

\[ \delta_t \Sigma - \Delta(t) = 0 \quad \Sigma \in L^2 \]

\[ \Rightarrow \delta_t \Sigma = \Delta(t) \Sigma \in L^2 \]

\[ \Rightarrow \Sigma \in L^2 \quad \text{with } \| \Sigma \| \leq C \| \Sigma \|_{L^2} \]

\[ \Rightarrow \Sigma \text{ smoothly} \]

(b) Claim: A seq of solutions $\Sigma$ on a finite time interval $\mathcal{I}$, $T$ such that $\Sigma$ bounded in $L^2$ converges in $C^0$. 

Pf: bounded in $L^2$ $\rightarrow$ 61 in $L^2$ $\times L^2$ by above

Rellich lemma $\Rightarrow$ on compact domain conv

\[ \Rightarrow \Sigma \text{ in } C^0 \text{ in } L^2_{k-1} \]
(63) We have a bound on "energy" \( \| \mathbf{E} \|_{L^2} \) via \( \int \| \nabla \mathbf{E} \|^2 = \) energy difference

\[
\text{and maybe (4) finite energy flow lines must come to cuto\'s} \quad \Rightarrow \quad \text{conv to broken flow line}
\]

Let's start with Fredholm index. Fix \( \Psi \) J-holo. We have \( \mathcal{E} \)
\[
\Psi(u) = \overline{\mathcal{G}}_{\xi} \mathcal{F} \in \mathcal{E}
\]

\[
L^p_k(\Sigma, \mathcal{M}) = L^p_k(\Sigma, \Lambda^{0,1} T^* \mathcal{M} \otimes u^* \mathcal{M})
\]

\[
\mathcal{D}_u := \nabla = \frac{\partial}{\partial u} : L^p_k(\Sigma, u^* \mathcal{T} \mathcal{M}) \rightarrow L^p_k(\Lambda^{0,1} T^* \Sigma \otimes u^* \mathcal{T} \mathcal{M})
\]

"linearization of \( \mathcal{D}_u \) at \( \mathcal{U} \)

\[
(\mathcal{D}_u \xi)(\frac{\partial}{\partial s}) = \frac{\partial}{\partial s} \xi + \oint (u(s,t)) \frac{\partial}{\partial t} \xi
\]

\[
+ \left( \frac{\partial}{\partial s} \xi(s,t) \right) \frac{\partial u}{\partial t}
\]

Similarly for \( \frac{\partial}{\partial t} \)

Thus:
\[
\mathcal{D}_u \xi = \nabla \xi + \text{0th order}
\]

\[
(\text{in coordinates}) \quad \Rightarrow \quad A(s,t) \xi(s,t)
\]
\[ \overline{\partial}_J f \neq 0 \text{ if } f \neq 0 \]  

Remark: may only work for real \( f \)

\[ \frac{\partial f}{\partial t} = f \overline{\partial}_J f \]

But:\ Claim: \( \overline{\partial}_J f = \overline{\partial}_J + B \)

This: \( D \) is a compact perturbation of a complex Cauchy-Riemann operator

\( I: D: \mathcal{L}^2(\Sigma,\nu^*TM) \rightarrow \mathcal{L}^2(\Sigma,\nu^*TM) \)

\( \text{satisfies hypotheses over } \Sigma \in C^\infty(\Sigma,\mathbb{C}) \)

Theorem (Riemann-Roch)

\( D \) is Fredholm, \( \ker D = H^0_{\overline{D}}(\Sigma,\nu^*TM) \)

\( \operatorname{cok} D = H^1_{\overline{D}}(\Sigma,\nu^*TM) \)

with \( h^0_D - h^1_D = n \chi(\Sigma) + 2 \varepsilon(E)[\Sigma] \)

\( \text{ind}(D) = n \cdot \dim \text{ of fibers} \) of \( \nu^*TM \)

Here's a sketch based on cutting into disks (see CMS2, Appendix C7) which also gives the index when we have Lagrangian \( J \) circles.
Let \( E \subseteq \Sigma \) be \( k \)-tuple with \( F = E^T \sigma \) (or: symp) \( k \) totally real \( \Sigma \) (or: flag).

We can define \( \mu(E,F) = \left[ Z, (E,F)[\Sigma] \right] \).

For \( E \) finite tuple this is \( 2^k \) tensor (with signs) w/ 
\( \Sigma \)-formable of a section which is 
\( \mu(E,F) \) over \( \Sigma \).

\[ \text{extend by additive under direct sum, under homotopy} \]
\( \lambda \) is \( \Sigma \) of indole pair.

**Notes:**

1. \[ \text{satisfies } \]
   \[ \mu(F_1,E) = \mu(F_2,E) + \mu(F_1,F_2) = \mu(E,F) \]

2. For \( Z = D^{n+1} \) disk \( c \in C \)
   \( E = D \times c \)
   \( F = e^{\frac{k}{n+1} \theta} \mathbb{R} \) (at \( e^{\frac{n}{n+1} \theta} \mathbb{R} \))
   \( \mu(E,F) = k \) (for \( k = 2 \), \( Z \) is a section \( \Sigma \) one, zero-

3. For \( \Sigma = \emptyset \), \( \mu(E,F) = 2 \sigma(E)[\Sigma] \)
Thus: Index of $D_n : \mathcal{L}_k(\Sigma, u^* TM, u^* TL)$
for $p_k > 1$ is

$$\pi \chi(\Sigma) + \mu(u^* TM, u^* TL)$$

Sketch: (1) We omit estimates showing $D_n$ and adjoint are Fredholm.

(2) Check for $D_n = \bar{D}$, $u^* TM = 0$, $\Sigma = 0$, $u^* TL = e^{i\theta}$.

Claim: If $k \leq -1$ then $D = m_j$; if $k > 0$ then $d = e^{i\theta}(k+1)$.

Proof: Let $D \in C^\infty$. Write $D(z) = \sum a_n z^n$.

\[ \mathcal{D}(\Sigma) = \sum a_n e^{i\theta} \] (former exp)

We have $a_n = a_{k-n}^j$.

Thus $D = a_0 + a_1 z + \cdots + a_k z^k$ with $k+1$ conditions, so $d = e^{i\theta}(k+1) = k+1$.

(3) General case is trickier than perturb. of this one.

(4) Giving Claim, modifies add under

Sketch: allow domain $\mathcal{M}$.

“Minimal layer $\Theta$ cards”:

Let $j: S^i \to \Sigma, j = 1 \ldots l$ being cph.
Form $i \circ j^* E$. This is as any
Layer subbundle of this. (Estimates show
Reduction still work)

Then giving is a "diagonal" subbundle

homotopic to $F_1 \circ F_2 \Rightarrow$ homotopic Frederick

$\Rightarrow$ same index.