Combinatorial Legendrian contact homology

Let \( \mathcal{L}_c(K) = \mathcal{H}/2 \langle \phi \mid \phi \text{ a crossing in } \pi(K) \rangle \)

free tensor algebra.

Let \( \mathcal{L}_{c, k}(p) \), \( q_i, -q_i \) be

\[
\sum_{\text{surgeries } (D^2, \partial D^2, x, y_1, y_k)} \frac{1}{y_1 + y_k} \rightarrow \mathcal{L}_{c, k}(p) \quad \text{taking } x \to \pm 1, \quad y_i \to -y_i \quad \text{up to isotopy}
\]

\[
\begin{array}{c}
\text{clockwise ordering.}
\end{array}
\]

\[
(*) \quad \partial \varphi = \sum_{k=1}^{k_{max}} \left( \pm \mathcal{L}(p, q_{k_{1}} \cdots - q_{k_k}) \right) q_{k_{1}} \cdots - q_{k_k}
\]

\[
\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots + \varphi_n \quad \text{mod 2}
\]

Thm (Chen, Legendrian contact homology) \((1)\) \( \partial^2 = 0 \)

\[
(2) \quad \mathcal{L}(K) \text{ is well-def up to } \text{ pseudo-isotopy (in fact, "stable framed isotopy")}
\]

Remark: finitely many terms in \((*)\):

Action \( A(p) = \text{height difference at crossing} \)

\[
(= \int \alpha \cdot \gamma \text{ Reeb chord at crossing})
\]
Claim: \[ A(p) - \Sigma A(q_i) = \int_{\mathbf{R}^3} dx \wedge dy \]

Cor: only fin. many terms \[ = \int_{\mathbf{R}^4} dx \]

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How do we extract info from \( LC_0(K) \)?

**Augmentation & Linearized Homology**

*Given a dga \( A \) over a field \( k \), an augmentation is a dga map \( \varepsilon: (A, \partial) \to (k, 0) \)

1. \( \varepsilon \) is \( k \)-alg map \( \implies \varepsilon(1) = 1 \) \( \circ \) \( \varepsilon \)

2. \( \varepsilon \circ \partial = 0 \implies \partial \circ \varepsilon = 0 \)

Example:

\( \varepsilon(p) = 0 \) for all crossing \( p \); \( \varepsilon \) is an augmentation for \( LC_0(K) \) \( \implies \partial_0 = 0 \).

Let \( I_\varepsilon = \ker \varepsilon \). Then \( \partial: I_\varepsilon \to I_\varepsilon \)

(and in fact \( (I_\varepsilon)^{\leq 0} \)) call this \( \varepsilon \)

Form \( H_{\ell,0}^\varepsilon(A) = H_0(I_\varepsilon/I_\varepsilon^2, \varepsilon) \).

Thus \( H_{\ell,0}^\varepsilon(LC_0) = H_{\ell,k}(\bigoplus_{n \geq 0} I_{\varepsilon} \otimes \mathbf{R}^2, \varepsilon) \)
Given $A \xrightarrow{\phi} B$ we get an argumentation
\[ \phi ; 3 \circ \phi : A \xrightarrow{?} \]

**Question:** Under what conditions does
\[ \phi \quad \text{ quasi-iso } \Rightarrow \phi^2 : H_{\circ \phi}(A) \xrightarrow{?} H_{\circ \phi}(B) \]

is an iso?

**Proof (Chekanov):** $A$, $B$ have free tensor products $V_k$ as their underlying algebras and $\phi$ is a "stable tame" iso then
\[ \phi^2 \text{ is an iso (exercise)} \]

**Tame Iso:** $\mathbb{Z}_2 < a_1 \rightarrow \mathbb{Q} \rightarrow \mathbb{Z}_2 < \theta_1, \theta_k >$

\[ a_i \mapsto b_0(i) \neq k \]
\[ a_k \mapsto b_0(k) + c \]
\[ c \in \mathbb{Z}_2 < \theta_1, \theta_k > \]

**Stable Iso:** $A \rightarrow A \otimes \mathbb{Z}_2 < a, b >$

inclusion or projection $(\exists a=0, \exists b=a)$
\[ (a, b \rightarrow 0) \]
Example calculation: $S_2$ is

\[ \partial p_5 = \ldots = \partial p_6 = 0 \]
\[ \partial p_1 = 1 + p_2 + p_7 p_5 \]
\[ \partial p_2 = 1 + p_9 + p_5 p_6 + p_9 \]
\[ \partial p_3 = 1 + p_9 p_7 \]
\[ \partial p_4 = 1 + p_8 p_9 \]

Any $m$: let $E(p_i) = c_i$. Then need

\[ 1 + c_7 + c_5 c_6 c_7 = 0 \]
\[ 1 + c_9 + c_5 c_6 c_9 = 0 \]
\[ 1 + c_8 c_7 = 0 \]
\[ 1 + c_8 c_9 = 0 \]
\[ \implies c_7 = c_8 = c_9 = 0 \]
\[ c_5 c_6 = 0 \]

\[
\begin{align*}
\mathbb{I}_x & \mid \mathbb{I}_x^2 = \\
\oplus_{p_i} & \cdot (p_i - c_i)
\end{align*}
\]

\[ \partial \tilde{p}_1 = \tilde{p}_2 + c_5 \tilde{p}_4 + c_6 \tilde{p}_5 \]
\[ \partial \tilde{p}_2 = \tilde{p}_9 + c_5 \tilde{p}_6 + c_6 \tilde{p}_5 \]
\[ \partial \tilde{p}_3 = \tilde{p}_9 + \tilde{p}_8 \]
\[ \partial \tilde{p}_4 = \tilde{p}_9 + \tilde{p}_1 \]

Exercise: work out the others.