Contact Structures:

\[ (M^{2n+1}, \xi^{2n}) \] smooth, \( \xi \) a "maximally non-integrable" hyperplane distribution, i.e. \( \xi_p \subset TM_p, \) hyperplane constrained \( \Rightarrow \exists \alpha \in \Omega^1(M) \) s.t. \( \ker \alpha = \xi_p, \forall p \) or \( \ker \alpha = \xi. \)

Note that for \( f : M \to \mathbb{R}_{>0} \) will also work

"maximally non-integrable": \( \alpha \wedge (d\alpha)^n \) is non-vanishing equicontinuously: \( (d\alpha)_p \) is symplectic on \( \xi_p. \)

Claim: Given \( L \subset M \) such that \( TL_p \cong \xi_p, \forall p \in L, \)
\[ \dim L \leq n \text{ (in fact } d\alpha|_L = 0 \}\]
Proof: \( \alpha|_L = 0. \) Thus \( 0 = d(\alpha|_L) \subset (d\alpha)|_L. \)

Such \( L \) are called isotropic. If \( \dim L = n, L \) is called Legendrian.

Two other points justifying "maximally non-integrable"
1) Legendrian submanifolds are in abundance, e.g. those through any \( p \) with \( TL_p \cong \mathbb{R}^n \subset \xi_p \)
2) Every hyperplane distribution admits \( n \)-dim integral submanifolds (omitted)

Remark: \( N \) integrable \( \Rightarrow \alpha \wedge (d\alpha)^n = 0 \) (compare Froeb. integrability)

Def: For \( \alpha \) non-oriented \( \alpha \wedge (d\alpha)^n > 0 \) positive contact\(^*\)
\( \alpha \wedge (d\alpha)^n < 0 \) negative contact\(^*\)
We'll mostly work in dim 3 with positive coordinates.

Examples: \( \mathbb{R}^3 \)
\[\alpha = dz - y dx \]
\[d\alpha = dx \wedge dy \]
\[\alpha \wedge d\alpha = dx \wedge dy \wedge dz\]

\(\mathbb{R}^3\)
\[\alpha = d\theta + \frac{1}{r} dr \wedge d\theta \]
\[d\alpha = r dr \wedge d\theta \]
\[\alpha \wedge d\alpha = r dr \wedge d\theta \wedge dz\]

2) \(\mathbb{R}^3\)
\[\alpha = d\theta + \frac{1}{r} dr \wedge d\theta \]
\[d\alpha = r dr \wedge d\theta \]
\[\alpha \wedge d\alpha = r dr \wedge d\theta \wedge dz\]

Same picture, but \(\theta\) and \(\phi\) invariant.

Exercise: There are two contactomorphisms, i.e.

\[\exists \varphi: (\mathbb{R}^3, \xi_0) \cong (\mathbb{R}^3, \xi_1)\]
\[\text{s.t. } \varphi(\xi_0) = \xi_1\]

3) \(\mathbb{R}^3\)
\[\alpha = \cos r \, dz + r \sin r \, d\theta \]
\[d\alpha = -\sin r \, dr \wedge dz + (\sin r \cos r) d\theta \]
\[\alpha \wedge d\alpha = \left(\cos^2 r \, r + \sin^2 r \cdot r + \sin r \cos r\right) dr \wedge dz \wedge d\theta \]
\[= \left(1 + \frac{\sin r \cos r}{r}\right) dx \wedge dy \wedge dz\]

This keeps twisting, thus an overtwisted disk:

Def: \((M, \xi)\) is overtwisted if \(\exists D \subset M\) s.t. \(TD_x = \xi_x \forall x \in \partial D\).

The disk of radius \(R/2\) is such in (3).
Thm: (1) has no amortized work (later).

4) Given $M$ smooth, $\tilde{T}^*M = \mathbb{R} \oplus TM$ has a natural contact structure with $\alpha = dz - h \eta \omega$

$$dx = dh \eta \omega = \omega \wedge \alpha \wedge (dx)^n = dz \wedge (\omega \wedge \alpha)^n$$

Example: $\tilde{T}^* \mathbb{R} = \mathbb{R}^3 \quad \alpha = dz - ydx$

5) (Contact hypersurfaces) Let $(M^n, \omega)$ be symplectic, noncompact. A Liouville vector field $\nu$ on $M$ satisfies

$$L_{\nu} \omega = \omega \quad \text{Suppose} \quad N \subseteq M \quad \text{is transverse to} \quad \nu \quad \text{Then Claim:} \quad (N, \ker i_{\nu} \omega)$ is contact

Proof: $L_{\nu} \omega = \frac{d}{ds} i_{\nu} \omega + i_{\nu} d\omega = 0$

$$\therefore \omega = dx \quad \text{so} \quad \omega \wedge (\omega \wedge \alpha)^{n-1} = \frac{1}{n} i_{\nu} \omega^n$$

$\omega^n \text{ volume form} \Rightarrow i_{\nu} \omega^n \text{ volume form on } N \text{ by } \nu.$

5') Let $M = T^*B \quad \omega = d\lambda$. Then define a Liouville vector field by $i_{\nu} \omega = \lambda$

$$(\lambda = \sum y_i dx_i, \quad \omega = \sum dx_i \wedge dy_i, \quad \eta = \sum y_i \frac{\partial}{\partial y_i})$$

Then $(T^*B, \lambda)$ is contact (transverse to $y = \text{radial}$)

5'') $S^3 \subseteq (\mathbb{R}^4, \omega)$. Let $\eta = \frac{\partial}{\partial r}$ radial v-field.

$$\alpha = \frac{i}{2} \left( \sum y_i dx_i - \sum x_i dy_i \right) \quad \eta = \frac{1}{2} \left( \sum x_i \frac{\partial}{\partial x_i} + \sum y_i \frac{\partial}{\partial y_i} \right)$$

$i_{\nu} \omega = \alpha \quad \text{so} \quad L_{\nu} \omega = dx = \omega$
$(S^3, \ker \alpha)$ contact.

Claim: $\ker \alpha|_{TS^3} = TS^3 \cap JTS^3$

Proof: Let $f(z) = |z|^2$, $f^{-1}(1) = S^3 - TS^3 = \ker \alpha|_{TS^3} = \ker (df|_J)$

$df \circ J = 4\alpha$.  

Reeb vector fields:

Prop: Given contact form $\alpha$, $J$! vector field $R_\alpha$ "Reeb vector field" such that:

(a) $i_{R_\alpha} \alpha = 0$

(b) $i_{R_\alpha} \alpha = 1$

Proof: $\ker \alpha$ on $TM$ is skew-symmetric of rank 2n. Thus has 1-dim kernel not contained in $\ker \alpha$.

Prop: Suppose $(N, \xi_c, (M, \omega))$ is a contact hypersurface with Liouville v.f. $N$, contact form $\alpha = \iota_N \omega$ and further no to the level set of a hamiltonian $H: M \to \mathbb{R}$. Then the Reeb flow is parallel to the Hamiltonian $H$.

Proof: $dH|_N$ has 1-dim kernel.

$i_{R_\alpha} \omega|_N = 0$ $i_{R_\alpha} \omega|_N = dH|_N = 0$

Thus $\{\text{periodic orbits of } R_\alpha\} \hookrightarrow \{\text{periodic orbits of } X_H\}$
Conjecture (Weinstein '70s): Let $N < R^{2n}$ be a hypersurface of contact type. Then $\exists$ closed characteristic (i.e., closed orbit of any $X_{\alpha}$, equiv of $R\alpha$).

Thm (Taubes 2007): Let $(N^3, \alpha)$ be a contact 3-mfld with choice of contact form $\alpha$. Then $\exists$ periodic orbit of $R\alpha$. 