1. Consider the symplectomorphism $\phi$ of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (a Dehn twist).

(a) Identify the set of fixed points of $\phi$.

(b) Exhibit a Hamiltonian isotopic map with only two fixed points.

(c) Exhibit a symplecically isotopic map with no fixed points.

(d) Show that every diffeomorphism smoothly isotopic to the map given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has at least one fixed point.

2. Let $L \subset (M^4, \omega)$ be a (framed) Lagrangian two-sphere in a symplectic 4-manifold. Let $\tau_L$ denote the Dehn twist about $L$. Show that $(\tau_L)^2$ is smoothly isotopic to the identity.

3. Consider $M^n_k = \{z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \ldots + z_{n-1}^2 + z_n^k = 1\}$, a 2n-dimensional symplectic (in fact Kähler) submanifold of $\mathbb{C}^{n+1}$. Consider the symplectic Lefschetz fibration $\pi: M^n_k \to \mathbb{C}$ given by $(z_0, \ldots, z_n) \to z_n$.

Remark: It is instructive throughout to restrict to the case $n = 1$, in which case the generic fiber consists of two points. You may wish to restrict your proofs to the cases $n = 1$ and $n = 2$.

(a) Show that for $M^2_2$ is symplectomorphic to $T^*S^n$. Show that $\pi: M^2_k \to \mathbb{C}$ is the pullback by the map $\mathbb{C} \to \mathbb{C}$ given by $w \mapsto 1 - w^k$ of the standard Lefschetz fibration from $\mathbb{C}^n$ to $\mathbb{C}$ with one singularity.

(b) Let $p$ and $q$ be distinct critical points of $\pi$. Show that matching cycles from $p$ to $q$ over isotopic paths $\gamma_1$ and $\gamma_2$ are symplectically isotopic.

Matching cycle: Consider a path $\gamma$ from one critical value to the other and a point $x$ on this path. Form two Lagrangian disks (Lefschetz thimbles) with boundary spheres contained in the fiber over $x$ by taking points over $\gamma$ to the left/right of $x$ limiting to the left/right (respectively) critical point under the horizontal flow over the path $\gamma$. Suppose (as is true in our case) the boundaries of these disks are isotopic in the fiber over $x$. Then after isotoping one of the disks and gluing it to the other disk, we obtain a Lagrangian sphere, our matching cycle.

(c) Let $p, q, r$ be distinct critical points of $\pi$. Let $L_{pq}$ be a matching cycle from $p$ to $q$ over a path $\gamma_{pq}$ in $\mathbb{C}$ from $\pi(p)$ to $\pi(q)$ avoiding all other critical values and $L_{qr}$ a matching cycle from $q$ to $r$ over a path $\gamma_{qr}$ from $\pi(q)$ to $\pi(r)$. Show that $\tau_{L_{qr}}L_{pq}$ is symplectically isotopic to a matching cycle $L_{pr}$ over the path $\gamma_{pr}$ given by concatenating $\gamma_{pq}$ and $\gamma_{qr}$ and pushing off the critical value $\pi(q)$ “to the right” (as viewed along the path from $\pi(p)$ to $\pi(r)$).