Dartouxs theorem: \((M, \omega)\) is locally diffeo to \((\mathbb{R}^{2n}, \omega_0)\).

Lemma:

\[
\frac{d}{dt} \left( \psi_t^* \omega_t \right) \bigg|_{t=t_0} = \psi_t^* \left( \frac{d\omega_t}{dt} \bigg|_{t=t_0} + \omega_t \right)
\]

(exercise)

Moser's trick:

Suppose we have a family of symplectic forms \(\omega_t\) on \(M\) with \([\omega_t] \in H^2(M)\) constant. That is \(\frac{d\omega_t}{dt}\) is exact = do_2

\[
\begin{array}{c}
\mathbb{R}^2 \\
\mathbb{Z}(M) \uparrow \text{const.} \downarrow \text{mod.} \downarrow \mathbb{Z}(M) \\
\mathbb{R}^2 \quad \mathbb{B}^2(M)
\end{array}
\]

Ask: can we find \(\psi_t: M \to M\), \(\psi_{t_0} = \text{id}\)

s.t. \(\psi_t^* \omega_t = \omega_{t_0}\)?

Connect to an ODE:

represent \(\psi_t\) as the flow of \(X_t\)

\[
\psi_t = X_{t_0} \psi_{t_0}, \quad \psi_{t_0} = \text{id}
\]
Then we ask \( \frac{d}{dt} \psi^* w_t = 0 \).

In terms of \( X_t \) (at time \( t \)):

\[
0 = \psi^* \left( \frac{d\omega_t}{dt} \bigg|_{t=T} + L X_T \omega_T \right)
\]

i.e.,

\[
0 = d\sigma_T + i x_t^* d\omega_T + d i x_T^* \omega_T
\]

\( q. \) we may let \( X_T = -\omega_T^{-1}(\sigma_T) \) (then \( i x_T \omega_T + \sigma_T = 0 \))

Now if \( M \) compact can now solve (2).

**Lemma:** Let \( M = Q \) compact subsets with \( w_0, \omega \), symplectic forms on \( M \).

Suppose \( (w_0)_t = (\omega)_t \) \( \forall \in \mathcal{Q} \) (i.e., on \( TM_Q \)).

Then \( \exists N_0(\mathcal{Q}), N_1(\mathcal{Q}) \) neighborhoods with

\[
\psi: N_0(\mathcal{Q}) \to N_1(\mathcal{Q}) \text{ diffeo, } \psi|_Q = \text{id}, \psi^* \omega_t = \omega_0
\]

**Cor.** Darboux's Theorem: Let \( \mathcal{Q} = \{pt\} \) and use prop above to get \( \omega_0 \) on \( TM_{pt} \).
Proof of Lemma: By Moser’s trick, it suffices to find $\sigma$ a $1$-form in $\mathcal{A}(\mathcal{W})$ s.t.

\[
\begin{align*}
\sigma(\tau M_1) &= 0 \\
\tau \sigma &= \omega - \omega_0 = \tau
d\tau &= \omega_0 + t d\omega
\end{align*}
\]

Then $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t d\omega$ is an exact deformation.

Shrink $\mathcal{W}(\lambda)$ if necessary s.t. $\omega_t$ is nondegenerate $\forall t$.

To find $\sigma$:

Claim (Generalized Poincare Lemma)

Let $\mathcal{E} \to \mathcal{B}$ be a smooth bundle, $\mathcal{E}$ a closed $k$-form vanishing on $\mathcal{B}$. Then $\exists$ $k$-form $\sigma$ vanishing on $\mathcal{B}$ defined on $U \supset \mathcal{B}$ with $d\sigma = \tau$.

Sketch: Run Poincare lemma on each fiber.

Let $h : \mathcal{U} \times [0,1] \to \mathcal{U}$ be a homotopy (not neces.
\[
\begin{align*}
& \text{between } id_{\mathcal{U}} \text{ and } \tau : \mathcal{U} \to \mathcal{B} \subset \mathcal{U}.
\end{align*}
\]

and let $j_t : \mathcal{U} \to \mathcal{U} \times [0,1]$ via $x \mapsto (x,t)$

Then let $\sigma = \int_0^1 j_t^* (i_{\partial_t} h^* \tau) \, dt$

("integration of $h^* \tau$ along fibers of $\mathcal{U} \times [0,1] \to \mathcal{U}\)

Check that this works (see [G1, App A] for details)!
**Neighborhood Theorems:**

**Theorem:** Let \( Q_0 \subset (M_0, \omega_0) \) and \( Q_1 \subset (M_1, \omega_1) \) be symplectic submanifolds of \( M_0, M_1 \).

Suppose \( \Phi \) is a symplectomorphism and \( \Phi \) is an isomorphism of symplectic vector bundles.

Then \( \Phi \) extends to \( (N(Q_0), \omega_0) \to (N(Q_1), \omega_1) \) such that \( d_\Phi \) induces \( \Phi : V_{Q_0} \to V_{Q_1} \).

**Proof:** Extend \( \Phi \) to a diffeomorphism and apply above lemma.

**Isotropic submanifolds:** \( L_0 \subset (M_0, \omega) \) with \( (TL)^* \omega = TL \).

Then \( TL = TM|_L / (TL)^* \oplus (TL)/TL \).  

**Lemma:** \( TM|_L / (TL)^* \cong TL \).  

**Proof:** \( X \to (i_x(\omega))_T \) well-defined inj, so too be of rank.

\( \square \)
Note: with $w$-compatible actions, we have

\[ \frac{\mathcal{L}}{\mathcal{H}} \xrightarrow{\alpha} \frac{\mathcal{L}}{\mathcal{H}} \xrightarrow{\xi} J(\mathcal{L}) \xrightarrow{\alpha} T\mathcal{L}/(T\mathcal{L}^*) \text{ at the inverse} \]

Lemma: \((\mathcal{L} \oplus J(\mathcal{L}), \omega) \rightarrow (\mathcal{L} \oplus T^*\mathcal{L}, \omega_{st})\)

is an iso of symplectic vector spaces \((\mathcal{L} \oplus J(\mathcal{L}))^* \rightarrow T^*(\mathcal{L})\)

Exercise, see [MS1, 3.3.7] or [GL, 2.5.7]

\(\left[ \eta_{ij} \right] \)

Then: Suppose \(L_1, L_2 \subset M_1, M_2\) as isotropic submanifolds and \(\exists \exists: \text{CSN}_{M_1}^*(L_1) \rightarrow \text{CSN}_{M_2}^*(L_2)\) is an iso of symplectic vector spaces \((\mathcal{L}_1, \omega_1) \rightarrow (\mathcal{L}_2, \omega_2)\).

Then \(q\) extends to a symplectomorphism of \(N(L_1) \rightarrow N(L_2)\).

Cor: \(L\) Lagrangian \(\Rightarrow N(L) \cong \text{nbhd of } L \text{ as zero set} \) in \((T^*L, \omega_{st})\).

Remarks on Lagrangians:

Given \(q: M \rightarrow M\) symplectomorphism,

\(\Gamma q = \{(x, q(x)) \mid x \in M \times \overline{M} \}
\neq \text{convex, } \omega q(\cdot, \cdot)\)

is Lagrangian.
Cor: A neighborhood of \( \text{id} \in \text{Sym}^\mathcal{L}(X) \) can be identified with a neighborhood of \( 0 \) in \( \mathbb{Z}^\ell(M) \).

\[ \begin{align*}
Pf: & \quad \Delta \subset M \times M \text{ is Lagrangian, has} \\
& \quad \text{nbd of } \text{nbd of } M \text{ in } T^*M \\
\end{align*} \]

Lagrangians near zero section are sections, \( \in \mathfrak{I} \)-forms. Condition on a \( 1 \)-form \( \alpha \) to be Lagrangian: \( d\alpha = 0 \)

\[ \begin{align*}
W = -d\lambda \text{can } \Rightarrow \quad & \quad \alpha: L \to T^*L \text{ is Lagrangian } \Leftrightarrow \\
0 = \alpha^* d\lambda \text{can} = d(\alpha^* \lambda \text{can}) \\
= & \quad d\alpha
\end{align*} \]

**Examples of symplectic manifolds:**

1. \( T^4 = (T^2 \times T^2, \omega \circ \omega) \)

2. \( T^2 \)-bundles over \( T^2 \):

   \( \text{SL}_2 \mathbb{Z} \) acts on \( T^2 \) by diffeomorphisms — in fact symplectic! Thus any map

\[ T_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \to \text{SL}_2 \mathbb{Z} \] gives a sympl. \( \text{nat}: \mathbb{R}^2 \times T^2 / (s \in \mathbb{Z}, t, x) \sim (t, s, t, A(x)) \)

with induced sympl.

\( (s, t, x) \sim (t, s, t, B(x)) \)
Exercise: build one with $rk H^1$ odd.  
(Cor: not Kähler.)

3. CP$n$ homogeneous.

On a chart where $z_j \neq 0$, set $z_j = 1$

$$\omega = \frac{i}{2} z \bar{z} \log \left( \sum_{i=0}^{n} z_i \bar{z}_i \right)$$