1. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$, and let $M \subset \mathbb{R}^n$ be given by a smooth level set of $f$; namely $df_p \neq 0$ for $p \in M$. For $X, Y \in TM_p \subset T\mathbb{R}^n_p$, show that

$$\langle S_N X, Y \rangle = \frac{-\text{Hess}(f)(X, Y)}{\langle \nabla f, \nabla f \rangle^{1/2}}.$$

Here $S_N X = -\nabla_X N$ for $N$ a unit normal. We choose $N$ to be the unit normal such that $df(N) > 0$. The Hessian is given by $\text{Hess}(f)(X, Y) = \nabla^*_X(df)(Y)$. (In this problem, all connections are the standard ones for $\mathbb{R}^n$.)

(b) Show that if the shape operator is positive or negative definite at $p$, then locally $M$ lies to one side (which?) of its tangent plane at $p$ (thought of inside $\mathbb{R}^n$). (Use problem 4 on problem set 1.)

2. (Gauss-Bonnet for contractible domains with piecewise smooth boundary) Consider $\mathbb{R}^2$ with an arbitrary metric $g$. Let $\nabla$ denote the Levi-Civita connection.

(a) Consider the differential equation $\frac{\partial}{\partial t} f(t) = B(t)f(t)$ for $f : \mathbb{R} \to \mathbb{R}^n$ and $B : \mathbb{R} \to \text{Mat}_{n \times n} \mathbb{R}$. Suppose that the $B(t)$ commute. Show that

$$f(t) = e^{\int_0^t B(t)dt} f(0)$$

is the unique solution with $f(0)$ given.\(^1\)

(b) Let $X$ and $Y$ be such that $X_p$ and $Y_p$ form an orthonormal basis of $T\mathbb{R}^2_p$ for each $p$. With respect to this basis, note that the connection is the trivial one plus $A$, an $\mathfrak{o}(2) \subset \mathfrak{gl}_2 \mathbb{R}$ valued 1-form,\(^2\) and the curvature is an $\mathfrak{o}(2)$-valued 2-form we denote $F_A$. Let $D$ be an embedding of a disk in $\mathbb{R}^2$ with piecewise smooth boundary. Show that $\int_{\partial D} A = \int_D F_A$, with $\partial D$ oriented counterclockwise.

(c) Show that the parallel transport counterclockwise around the boundary of $D$ is given, with respect to the orthonormal trivialization, as $e^{\int_{\partial D} F_A}$.

(d) Let

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Compute $e^{tB}$. Let $\psi : \mathfrak{o}(2) \to \mathbb{R}$ be the map taking $B$ to 1.

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\(^1\)There is a trick to define a variant of the exponential of the integral, called the path-ordered exponential, when they don’t commute.

\(^2\)Here we use the trivialization coming from the chosen orthonormal basis, not from $\mathbb{R}^2$.\(^3\)
(e) Suppose the boundary of $D$ is piecewise geodesic. Let $V$ be the set of vertices (i.e. places where the boundary is not smooth) and let $\alpha(v)$, for $v \in V$, be the “exterior angle” at $v$. That is, the amount in $[0, 2\pi)$ one has to rotate the tangent vector counterclockwise at $v$ to go from one piece of $\partial D$ to the next. Show that $2\pi = \sum_v \alpha(v) + \psi(\int_D F_A)$.

(f) Keep the notation of part (e) but drop the assumption that the boundary of $D$ is piecewise geodesic. Let $\kappa : \partial D \rightarrow \mathbb{R}$, smooth away from the vertices, give the signed curvature of the boundary. That is, letting $T$ be the unit tangent to the boundary and $N$ the unit normal pointing inward, $\kappa = \langle \nabla_T T, N \rangle$. Show that $2\pi = \sum_v \alpha(v) + \int_{\partial D} \kappa \, ds + \psi(\int_D F_A)$. Here $s$ parametrizes $\partial D$ with respect to arc length.

3. Let $\nabla$ be a connection on a vector bundle $E$ over a smooth manifold $M$. Suppose $X$ and $Y$ are two commuting vector fields on $M$. That is, $[X, Y] = 0$. Let $\phi^X_t : \Sigma \rightarrow \Sigma$ be the flow of $X$ at time $t$, and similarly for $\phi^Y_t$. Let $\Phi^\nabla_{\phi^X_t, \phi^Y_t}$ be the parallel transport for the connection $\nabla$ along the curve $\gamma(t)$ for time $s$. Let $Z_p \in TM_p$ be a vector at $p$.

Parallel transport $Z_p$ to

$V(s) = \Phi^\nabla_{\phi^X_t(\phi^Y_s(p)), s} \circ \Phi^\nabla_{\phi^Y_t(\phi^X_s(p)), s} Z_p \in T_{\phi^X_s(\phi^Y_s(p))} \Sigma = T_{\phi^X(\phi^Y(p))} \Sigma$

and then take it back around the other two edges of the “square” to

$W(s) = \left( \Phi^\nabla_{\phi^Y_s(\phi^X(p)), s} \circ \Phi^\nabla_{\phi^X_s(\phi^Y(p)), s} \right)^{-1} V(s) \in T_p \Sigma$.

Show that the difference $W(s) - Z_p = s^2 F_{\nabla}(X, Y) Z_p + s^3 h(s)$, where $h(s)$ is a smooth $TM_p$-valued function of $s$.

**Hint**: Mimic the sketch given in lecture that $t^2([X, Y]f)(p) + t^3 h(t) = f(p(t)) - f(p)$, where $p(t)$ is the point given by starting at $p$ and then flowing by $X$ then by $Y$ then by $-X$ then by $-Y$ each for time $t$. 